MATH GR6403 - Modern Geometry Notes

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1 Preliminaries

1.1 Immersions and Submersions

Let $f : M \to N$ be a smooth map between two manifolds. Then consider the differential map $df_p : T_pM \to T_{f(p)}N$.

Definition 1.1. Given a map f as defined above, we say that f is a immersion (submersion) if the differential df_p is an injective (surjective) linear map. This is also equivalent to the following:

Given $p \in M$, f is an immersion (submersion) at p if there is a chart (U, ϕ) for M around p and a chart (V, ψ) for N around $f(p)$ such that:

(i) $f(U) \subset V$

(ii) the composition $g = \psi \circ f \circ \phi^-1 : \phi(U) \to \psi(V)$ is an immersion (submersion) at $\phi(p)$

Definition 1.2. Let $f : M \to N$ be a smooth map between manifolds of dimension m and n respectively.

If f is an immersion (submersion) at $p \in M$, so that $m \leq n$ ($m \geq n$), there is a chart (U, ϕ) for M around p, and a chart (V, ψ) for N around $f(p)$ such that:

1. $\phi(p) = 0 \in \mathbb{R}^m$
2. $\psi(f(p)) = 0 \in \mathbb{R}^n$
3. The composition $\psi \circ f \circ \phi^-1$ is the restriction of the canonical immersion (submersion) to $\phi(U) \subset \mathbb{R}^m$

Additionally, if f is both an immersion and a submersion at p, then we call f a local diffeomorphism at p.

Definition 1.3. Let $f : M \to N$ be a smooth map between smooth manifolds. We say that f is an embedding if:

1. f is an immersion
2. $f : M \to f(M)$ is a homeomorphism onto $f(M)$, where $f(M)$ is equipped with the subspace topology.

In this case, we say that $f(M)$ is a submanifold of N.

1.2 Connections on a Vector Bundle

If E, F are smooth vector bundles over M, and $\phi : C^\infty(M, E) \to C^\infty(M, F)$ is a $C^\infty(M)$ linear map, then $\phi \in C^\infty(M, E^* \otimes F)$. For $s \in C^\infty(M, E)$, $\phi(s) \in C^\infty(M, F)$, and $\forall p \in M$, $\phi(p)(s) \in F$.

That is to say that $\phi(p) \in (E^* \otimes F)_p = \mathrm{Hom}_{\mathbb{R}}(E_p, F_p)$.

Let $\pi : M \to E$ be a C^∞ vector bundle over M. A connection ∇ on E is an $\mathbb{R}$-bilinear map:

$$\nabla : \mathfrak{X}(M) \times C^\infty(M, E) \to C^\infty(M, E)$$

which sends $(x, s) \mapsto \nabla_X s$ such that for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, and $s \in C^\infty(M, E)$, we have:

(i) $\nabla_{fX} s = f \nabla_X s$

(ii) $\nabla_X(fs) = X(f)s + f\nabla_X s$

Notice that for a fixed s:

$$\nabla_{\_} s \in C^\infty(M, T^*M \otimes E) = \Omega^1(M, E)$$

We will use the notation that $\Omega^k(M, E) := C^\infty(M, \Lambda^k T^*M \otimes E)$ to denote the space of E-valued k-forms on M. On the other hand, for a fixed X, $\nabla_X : C^\infty(M, E) \to C^\infty(M, E)$ is a derivation.

Alternatively, a connection ∇ on E can be viewed as an $\mathbb{R}$-linear map $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ that obeys the rule $\nabla(fs) = df \otimes s + f\nabla s$

The space of all connections on E is an infinite dimensional affine space whose associated vector space is $\Omega^1(M, \mathrm{End}(E))$

1.3 Pullback Bundles

Let $f : M \to N$ be a smooth map, and let $\pi : E \to N$ be a smooth vector bundle on N. Then we can define a bundle $\tilde{\pi} : f^*E \to M$ called the *pullback bundle* in the following way. As a set,

$$f^*E = \bigcup_{p \in M} E_{f(p)} = \{(p, q) \in M \times E \mid f(p) = \pi(q)\}$$

We can define a smooth structure on this in the following way. If $s : N \to E$ is a smooth section of E, then $f^*s : M \to f^*E$ given by:

$$f^*s(p) = s(f(p)) \in E_{f(p)} =: (f^*E)_p$$

is a smooth section of f^*E. If $e_1, \ldots, e_r$ are a smooth frame for $E|_U$, where U is an open set in N, then $f^*e_1, \ldots, f^*e_r$ is a smooth frame for $f^*E|_{f^{-1}(U)}$. A section $s : f^{-1}(U) \to f^*E|_{f^{-1}(U)}$ is smooth if and only if we can write:

$$s = \sum_{j=1}^{r} a_j f^* e_j$$

where the a_j are smooth functions of $f^{-1}(U)$. We then also have a pullback map

$$f^* : C^\infty(N, E) \to C^\infty(M, f^*E)$$

Suppose that $\{U \mid \alpha \in I\}$ is an open cover of N with local trivializations $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$, and define the transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ as before. Then,

$$f^*t_{\alpha\beta} = t_{\alpha\beta} \circ f : f^{-1}(U_\alpha \cap U_\beta) = f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to GL(r, \mathbb{R})$$

are the transition functions for the pullback bundle f^*E.

1.4 Pullback Connection

Let $f : M \to N$ be a smooth map, and let $\pi : E \to N$ be a smooth vector bundle with a connection ∇. Then there is a unique connection $f^*\nabla$ on f^*E called the *pullback connection* such that:

$$(f^*\nabla)(f^*s) = f^*(\nabla s)$$

for a smooth section $s : N \to E$.

In other words, if $s : N \to E$ is a smooth section, and $p \in M$, $X \in T_pM$, then

$$(f^*\nabla)_X (f^*s) = f^*\left(\nabla_{df_p(X)} s\right)$$

In terms of local coordinates, if $e_1, \ldots, e_r$ is a smooth frame for $E|_U$, then $f^*e_1, \ldots, f^*e_r$ is a smooth frame for $f^*E|_{f^{-1}(U)}$. On U, we know that:

$$\nabla e_j = \sum_{k=1}^{r} \omega_j^k \otimes e_k$$

Then

$$(f^*\nabla)(f^*e_j) = f^*(\nabla e_j) = \sum_{k=1}^{r} f^*(\omega_j^k) \otimes f^*e_k$$

Therefore, if $\{\omega_\alpha \in \Omega_1(U_\alpha, \mathfrak{gl}(r,\mathbb{R})) \mid \alpha \in I\}$ are connection 1-forms of the connection ∇ on $E \to N$, then $\{f^*\omega_\alpha \in \Omega_1(f^{-1}(U_\alpha), \mathfrak{gl}(r,\mathbb{R})) \mid \alpha \in I\}$ are the connection 1-forms of the pullback connection $f^*\nabla$ on $f^*E \to M$.

An important special case of this is if $E = TN$, with $f^*TN = TM$.

We then get a map $f_* : \mathfrak{X}(M) \to C^\infty(M, f^*TN) \leftarrow f^*\mathfrak{X}(M)$

With this map, we can say that X and Y are f-related if and only if $f^*Y = f_*X$ in $C^\infty(M, f^*TN)$.

And then given a connection ∇ on a vector bundle $\pi : E \to M$, we define for all $X, Y \in \mathfrak{X}(M)$:

$$R_\nabla(X,Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

Thus $R_\nabla \in \Omega^2(M, \mathrm{End}(E))$

And note that $R_{f^*\nabla} = f^*(R_\nabla)$

1.5 Derivative of Metric

Consider a Riemannian manifold (M, g) with the metric g_{ij} defined such that $g_{ij} = e_i \cdot e_j$. We can compute its derivative in the following way:

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k} g_{ij} \\
&= \frac{\partial}{\partial x^k} \left(e_i \cdot e_j\right) \\
&= \frac{\partial e_i}{\partial x^k} \cdot e_j + e_i \cdot \frac{\partial e_j}{\partial x^k} \\
&= \Gamma^\lambda_{ik} e_\lambda \cdot e_j + e_i \cdot \Gamma^\lambda_{jk} e_\lambda \\
&= \Gamma^\lambda_{ik} g_{\lambda j} + \Gamma^\lambda_{jk} g_{i\lambda}
\end{aligned}$$

So we have the following result:

$$\partial_k g_{ij} = \Gamma^\lambda_{ik} g_{\lambda j} + \Gamma^\lambda_{jk} g_{i\lambda}$$

1.6 Musical Isomorphisms

Let (M, g) be a Riemannian manifold, and suppose we choose coordinates $(x_1, \ldots, x_n)$ around $p \in M$ so that $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ is an orthonormal frame for T_pM. Then $\left(dx^1, \ldots, dx^n\right)$ is the dual frame for T_p^*M.

Then we can define the *musical isomorphism* operators $\flat$ and $\sharp$ in the following way:

$$\begin{aligned} \flat : T_pM &\to T_p^*M \\ X &\mapsto g_{ij}X^i dx^j \\ &\mapsto X_j dx^j \end{aligned} \qquad \begin{aligned} \sharp : T_p^*M &\to T_pM \\ \omega &\mapsto g^{ij}\omega_i \frac{\partial}{\partial x^j} \\ &\mapsto \omega^i \frac{\partial}{\partial x^i} \end{aligned}$$

Which gives us the relation $\langle \omega^\sharp, Y\rangle = \omega(Y)$

1.7 Gradient, Divergence, and Laplacian

Let (M, g) be a Riemannian manifold, and let ∇ be the Levi-Civita connection on (M, g). Given a vector field $Y \in \mathfrak{X}(M)$, we can write $Y = Y^i \frac{\partial}{\partial x^i}$ in a coordinate neighborhood U with local coordinates $(x_1, \ldots, x_n)$, where $Y^i \in C^\infty(M)$

Then, we have that:

$$\begin{aligned} \nabla_i Y &= \nabla_i Y^j \partial_j \\ &= \frac{\partial Y^j}{\partial x^i}\partial_j + Y^j \nabla_i \partial_j \\ &= \frac{\partial Y^j}{\partial x^i}\partial_j + Y^j \Gamma^k_{ij}\partial_k \\ &= \frac{\partial Y^j}{\partial x^i}\partial_j + Y^k \Gamma^j_{ik}\partial_j \\ &= \frac{\partial Y^j}{\partial x^i}\partial_j + \Gamma^j_{ik} Y^k \partial_j \end{aligned}$$

Which implies that:

$$\nabla_i Y^j = \frac{\partial Y^j}{\partial x^i} + \Gamma^j_{ik} Y^k$$

1.7.1 Gradient

Proposition 1.4. Given a Riemannian manifold (M, g), the gradient of a smooth function $f \in C^\infty(M)$ is given by:

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$$

Proof:

For any smooth function $f \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$ we can define a smooth vector field $\operatorname{grad} f \in \mathfrak{X}(M)$ by the rule:

$$\langle \operatorname{grad} f, X\rangle = df(X)$$

Note that this makes sense if we consider what both the gradient and derivative operators do in $\mathbb{R}^n$.

In local coordinates, we can write:

$$\nabla f = \left\langle \frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right\rangle \in \mathfrak{X}(M)$$

And also:

$$df = \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(M)$$

In this form, we can see that the components of both df and ∇f are the same.

So taking inspiration from Section 1.6 on musical isomorphisms, we can see that the definition can be rewritten as:

$$\langle df^\sharp, X \rangle = df(X)$$

Meaning that indeed the gradient and differential are related via $df^\sharp = \operatorname{grad} f$

$$df^\sharp = \operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

Therefore, we have shown:

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \tag{1.1}$$

□

1.7.2 Divergence

Now, let's consider the divergence of a vector field Y:

Proposition 1.5. Given a vector field $Y \in \mathfrak{X}(M)$, the divergence of Y is given by:

$$\operatorname{div} Y = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} Y^i \right)$$

Proof:

$$\begin{aligned} \operatorname{div}(Y) &= \nabla_i Y^i \\ &= \frac{\partial Y^i}{\partial x^i} + \Gamma^i_{ik} Y^k \end{aligned}$$

Now let us calculate Γ^i_{ik} in local coordinates:

$$\begin{aligned} \Gamma^i_{ik} &= \frac{1}{2} g^{ij} \left(\partial_i g_{kj} + \partial_k g_{ji} - \partial_j g_{ik} \right) \\ &= \frac{1}{2} g^{ij} \partial_k g_{ij} \\ &= \frac{1}{2} \operatorname{Tr} \left(g^{-1} \partial_k g \right) \\ &= \frac{\partial}{\partial x^k} \log \left(\sqrt{\det(g)} \right) \\ &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^k} \sqrt{\det(g)} \end{aligned}$$

We can plug this back into the expression for divergence, to get:

$$\begin{aligned}
\operatorname{div} Y &= \frac{\partial Y^i}{\partial x^i} + \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^k} \left(\sqrt{\det(g)} \right) Y^k \\
&= \frac{1}{\sqrt{\det(g)}} \left(\sqrt{\det(g)} \frac{\partial Y^i}{\partial x^i} + \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} Y^i \right) \right) \\
&= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} Y^i \right)
\end{aligned}$$

So we arrive at the formula:

$$\operatorname{div} Y = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} Y^i \right) \tag{1.2}$$

□

1.7.3 Laplacian

Proposition 1.6. Given a smooth function $f \in C^\infty(M)$, the Laplacian of f is given by:

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} g^{ij} \frac{\partial f}{\partial x^j} \right)$$

Proof:

The expression for the Laplacian in local coordinates follows quite trivially from the previous sections. We know that $\Delta f = \operatorname{div}(\operatorname{grad} f)$, so we can write:

$$\begin{aligned}
\Delta f &= \operatorname{div} (\operatorname{grad} f) \\
&= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} (\operatorname{grad} f)^i \right) \\
&= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} g^{ij} \frac{\partial f}{\partial x^j} \right)
\end{aligned}$$

Which gives the final result:

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} g^{ij} \frac{\partial f}{\partial x^j} \right) \tag{1.3}$$

□

2 Jacobi Fields

Let (M,g) be a Riemannian manifold. A Jacobi field $J(t)$ along a geodesic $\gamma : I \to M$ is a smooth vector field which is defined in the following way:

Consider a smooth map

$$f : (-\epsilon, \epsilon) \times [0,a] \to M$$
$$(s,t) \mapsto f_s(t) = f(s,t)$$

And we think of this as a family of geodesics parameterized by $s \in (-\epsilon, \epsilon)$ such that for any $s \in (-\epsilon, \epsilon)$, $f_s : [0,a] \to M$ is a geodesic with $f_0 = \gamma$.

We then define:

$$J(t) = \frac{\partial f}{\partial s}(0,t)$$

Lemma 2.1. *Let $A = (-\epsilon, \epsilon) \times [0,a] \subset \mathbb{R}^2$. Let $f : A \to M$ be any smooth map.Then $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are global smooth vector fields on A. Remember that we defined:*

$$\frac{\partial f}{\partial s} := f_* \left(\frac{\partial}{\partial s} \right), \quad \frac{\partial f}{\partial t} := f_* \left(\frac{\partial}{\partial t} \right) \in C^\infty \left(A, f^*TM \right)$$

Suppose that ∇ is the Levi-Civita connection on (M,g). Let $D = f^*\nabla$ be the pullback connection on f^*TM. Then:

$$\frac{D}{\partial s} \left(\frac{\partial f}{\partial t} \right) - \frac{D}{\partial t} \left(\frac{\partial f}{\partial s} \right) = 0 \tag{2.1}$$

$$\frac{D^2}{\partial t^2} \frac{\partial f}{\partial s} - \frac{D}{\partial s} \left(\frac{D}{\partial t} \frac{\partial f}{\partial t} \right) + R \left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) \frac{\partial f}{\partial t} = 0 \tag{2.2}$$

Proof of Lemma 2.1

First, we will prove equation 2.1.

By the symmetry of the pullback connection, we know that $f_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0$. Then:

$$0 = f_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = f_* \left(\frac{\partial}{\partial s} \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \frac{\partial}{\partial s} \right) = D_{\frac{\partial}{\partial s}} f_* \frac{\partial}{\partial t} - D_{\frac{\partial}{\partial t}} f_* \frac{\partial}{\partial s}$$

Which can be easily rewritten as equaiton 2.1.

Now to prove equation 2.2:

Remember that the Riemann curvature tensor R is defined as:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.3}$$

Then from this equation, we can see that the pullback of the curvature tensor can be written as:

$$R \left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right) f_* \left(\frac{\partial}{\partial t} \right) = D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial s}} f_* \frac{\partial}{\partial t} - D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} f_* \frac{\partial}{\partial t} - D_{\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right]} f_* \frac{\partial}{\partial t}$$

But we know that $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$, so the last term in the above equation is 0. Then we can rewrite the above equation as:

$$R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) f_*\left(\frac{\partial}{\partial t}\right) = D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial s}} f_* \frac{\partial}{\partial t} - D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} f_* \frac{\partial}{\partial t}$$

Referring back to equation 2.1:

$$\frac{D}{\partial s}\left(\frac{\partial f}{\partial t}\right) - \frac{D}{\partial t}\left(\frac{\partial f}{\partial s}\right) = 0 \implies \frac{D}{\partial s}\left(\frac{\partial f}{\partial t}\right) = \frac{D}{\partial t}\left(\frac{\partial f}{\partial s}\right)$$

So we can swap the order of differentiation to get:

$$R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\left(\frac{\partial f}{\partial t}\right) = D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} - D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial t}$$

And we can see that with some simple rearranging, and using $R(X,Y)Z = -R(Y,X)Z$ we get:

$$\frac{D^2}{\partial t^2}\frac{\partial f}{\partial s} - \frac{D}{\partial s}\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}\right) + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = 0$$

Which is precisely equation 2.2.

2.1 Jacobi Equation

Now, by the defining property of a geodesic, given $s \in (-\epsilon, \epsilon)$, we can see that any geodesic $f_s : [0, a] \to M$ as defined above must necessarily satisfy:

$$\frac{D}{\partial t}\frac{\partial f}{\partial t}(s,t) = 0 \quad \text{for any } s, t$$

Which lets us rewrite the equation again as:

$$\begin{aligned} R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\left(\frac{\partial f}{\partial t}\right) &= \frac{D^2}{\partial t^2}\frac{\partial f}{\partial s} \\ \frac{D^2}{\partial t^2}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\left(\frac{\partial f}{\partial t}\right) &= 0 \\ \frac{D^2}{\partial t^2}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial t}\right) &= 0 \end{aligned}$$

Notice that the sign changes due to the identity $R(X,Y)Z = -R(Y,X)Z$.

In particular, if we consider $s = 0$, and set:

$$\frac{\partial f}{\partial t}(0,t) = \gamma'(t) \quad \text{and} \quad \frac{\partial f}{\partial s}(0,t) = J(t)$$

Then we get the Jacobi Equation:

$$\frac{D^2}{\partial t^2}J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0 \tag{2.4}$$

Definition 2.1. A vector field $J(t)$ along a geodesic $\gamma : [0,a] \to M$ is called a *Jacobi field* if it satisfies the Jacobi Equation (2.4).

Definition 2.2. Let $\gamma : [0,a] \to M$ be a geodesic on a manifold M, with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$, so that $\gamma(t) = \exp_p(tv)$. Then:

(a) For any $u, w \in T_pM$, there is a unique Jacobi field $J(t)$ along $\gamma(t)$ with $J(0) = u$ and $\frac{DJ}{\partial t}(0) = w$.

(b) If $J(t)$ is a Jacobi field along $\gamma(t)$, then there is a smooth map $f : (-\epsilon, \epsilon) \times [0,a] \to M$ written $f(s,t) = f_s(t)$ such that:

  (i) for each $s \in (-\epsilon, \epsilon)$, $f_s : [0,a] \to M$ is a geodesic.

  (ii) $f_0(t) = \gamma(t)$

  (iii) $\frac{\partial f}{\partial s}(0,t) = J(t)$

Definition 2.3. Let $\gamma : [0,a] \to M$ be a geodesic on a manifold M, with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$, so that $\gamma(t) = \exp_p(tv)$. Also let $J(t)$ be a Jacobi field along $\gamma(t)$ such that $J(0) = 0$ and $\frac{DJ}{\partial t}(0) = w$. Then for $t \in [0,a]$:

$$J(t) = (d\exp_p)_{tv}(tw)$$

Lemma 2.2. *Let $\gamma : [0,a] \to M$ be a geodesic and $J(t)$ a Jacobi field along $\gamma(t)$. Then:*

$$\langle J(t), \gamma'(t)\rangle = \langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle$$

Proof

Define a smooth function $f : [0,a] \to \mathbb{R}$ by $f(t) = \langle J(t), \gamma'(t)\rangle$. The lemma can then be restated as $f(t) = f(0) + tf'(0)$. It suffices to show that $f''(0) = 0$.

Remember that because γ is a geodesic, $\frac{D}{dt}\gamma'(t) = 0$. Then:

$$\begin{aligned}
f'(t) &= \langle J'(t), \gamma'(t)\rangle + \langle J(t), \gamma''(t)\rangle = \langle J'(t), \gamma'(t)\rangle \\
f''(t) &= \langle J''(t), \gamma'(t)\rangle + \langle J'(t), \gamma'(t)\rangle = \langle J''(t), \gamma'(t)\rangle \\
&= \langle J''(t), \gamma'(t)\rangle = -\langle R(J(t), \gamma'(t))\gamma'(t), \gamma'(t)\rangle \\
&= R(J(t), \gamma'(t), \gamma'(t), \gamma'(t)) = 0
\end{aligned}$$

Remark 2.3. Note that both $\gamma'(t)$ and $t\gamma'(t)$ are Jacobi fields along the geodesic γ. Then by the previous lemma, we see that:

$$J(t) = \left(\langle J(0), \gamma'(0)\rangle\right) + t\left(\langle J'(t), \gamma'(t)\rangle\right)\frac{\gamma'(t)}{|\gamma'(0)|^2} + J^{\perp}(t)$$

Where $J^{\perp}(t)$ is also a Jacobi field along γ and

$$\langle J^{\perp}(t), \gamma'(t)\rangle = 0$$

2.2 Jacobi Fields on Manifolds with Constant Sectional Curvature

Suppose (M, g) is a Riemannian manifold with constant sectional curvature K. Let $\gamma : [0, a] \to M$ be a *normalized geodesic* ($|\gamma'|^2 = 1$). Let $\gamma(0) = p \in M$, and $\gamma'(0) = v \in T_pM$. Then let $J(t)$ be a Jacobi field along $\gamma(t)$ such that:

$$J(0) = 0, \quad \frac{DJ}{\partial t}(0) = w, \quad \langle w, v \rangle = 0$$

Then $\langle J(t), \gamma'(t) \rangle = 0$ for all $t \in [0, a]$. For any smooth vector field $V(t)$ along $\gamma(t)$:

$$\langle R(J, \gamma')\gamma', V \rangle = K \left(\langle \gamma', \gamma' \rangle \langle J, V \rangle - \langle \gamma', V \rangle \langle \gamma', J \rangle \right) = \langle KJ, V \rangle$$

Therefore, $R(J, \gamma')\gamma' = KJ$, so J satisfies:

$$\frac{D^2 J}{dt^2} + KJ = 0$$

Let $J(t) = f(t)w(t)$ where f is a smooth function on $[0, a]$, and $w(t)$ is the unique parallel vector field along $\gamma(t)$ such that $w(0) = w$. Then:

$$\frac{D^2 J}{dt^2} + KJ = 0, \quad J(0) = 0, \quad \frac{DJ}{dt}(0) = w$$

which is equivalent to:

$$f'' + Kf = 0, \quad f(0) = 0, \quad f'(0) = 1$$

Solving this differential equation, we get the solution:

$$f(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}}, & K > 0; \\ t, & K = 0; \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}}, & K < 0. \end{cases}$$

Therefore, the unique Jacobi field $J(t)$ along $\gamma(t)$ such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w$, where $\langle w, \gamma'(0) \rangle = 0$, is given by:

$$f(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}} w(t), & K > 0; \\ t w(t), & K = 0; \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} w(t), & K < 0. \end{cases}$$

Similarly, the unique Jacobi field $J(t)$ along $\gamma(t)$ such that $J(0) = u$ and $\frac{DJ}{dt}(0) = 0$, where $\langle u, \gamma'(0) \rangle = 0$, and $u(0) = u$ is given by:

$$J(t) = \begin{cases} \cos(\sqrt{K}t) u(t), & K > 0, \\ u(t), & K = 0, \\ \cosh(\sqrt{-K}t) u(t), & K < 0, \end{cases}$$

2.3 Taylor Expansion of g_{ij} in Local Coordinates

First, let us consider a geodesic $\gamma : [0, a] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, so that $\gamma(t) = \exp_p(tv)$. Also let $J(t)$ be a Jacobi field along this geodesic $\gamma(t)$ with $J(0) = 0$ and $\frac{DJ}{dt} = w \in T_pM$. Alternatively, this means that $J(t) = (d\exp_p)_{tv}(tw)$.

Now, let $f = \langle J, J\rangle$. We want to compute the taylor expansion of f in order to determine the taylor series for $\langle J, J\rangle = |J(t)|^2$.

By the product rule, we can see that:

$$\begin{aligned} f' &= \frac{D}{dt}\langle J, J\rangle \\ &= \langle \frac{DJ}{dt}, J\rangle + \langle J, \frac{DJ}{dt}\rangle \\ &= \langle J', J\rangle + \langle J, J'\rangle \\ &= 2\langle J', J\rangle \end{aligned}$$

Additionally:

$$\begin{aligned} f'' &= \frac{D}{dt} f' \\ &= \frac{D}{dt} 2\langle J', J\rangle \\ &= 2\langle \frac{DJ'}{dt}, J\rangle + 2\langle J', \frac{DJ}{dt}\rangle \\ &= 2\langle J'', J\rangle + 2\langle J', J'\rangle \end{aligned}$$

Continuing along with this pattern and repeatedly applying the necessary product rules to this function, we can see that we have the following table:

$$\begin{aligned} f' &= 2\langle J', J\rangle \\ f'' &= 2\langle J'', J\rangle + 2\langle J', J'\rangle \\ f^{(3)} &= 2\langle J^{(3)}, J\rangle + 6\langle J'', J'\rangle \\ f^{(4)} &= 2\langle J^{(4)}, J\rangle + 8\langle J^{(3)}, J'\rangle + 6\langle J'', J''\rangle \\ f^{(5)} &= 2\langle J^{(5)}, J\rangle + 10\langle J^{(4)}, J'\rangle + 20\langle J^{(3)}, J''\rangle \\ f^{(6)} &= 2\langle J^{(6)}, J\rangle + 12\langle J^{(5)}, J'\rangle + 30\langle J^{(4)}, J''\rangle + 20\langle J^{(3)}, J^{(3)}\rangle. \end{aligned}$$

Now we also need to compute the derivatives of $J(t)$ evaluated at 0. We already know that $J(0) = 0$ and $J'(0) = w$. We also can deduce from the Jacobi Equation that:

$$\begin{aligned} J'' + R(\gamma', J)\gamma' &= 0 \\ J''(0) + R(\gamma'(0), J(0))\gamma'(0) &= 0 \\ J''(0) + R(v, 0)v &= 0 \\ J''(0) &= 0 \end{aligned}$$

To compute the second derivative, we can simply take the fact that we know $J'' = -R(\gamma', J)\gamma'$, and differentiate both sides, giving us the following. Keep in mind that since $\gamma(t)$ is a geodesic, $\gamma''(t) = 0$ for any t.

$$\begin{aligned} J''' &= -R'(\gamma', J)\gamma' - R(\gamma'', J)\gamma' - R(\gamma', J')\gamma' - R(\gamma', J)\gamma'' \\ &= -R'(\gamma', J)\gamma' - R(\gamma', J')\gamma' \\ J^{(3)}(0) &= -R'(v, 0)v - R(v, w)v \\ &= -R(v, w)v \end{aligned}$$

And then for $J^{(4)}$, we can differentiate both sides again:

$$\begin{aligned} J^{(4)} &= -R''(\gamma', J)\gamma' - R'(\gamma'', J)\gamma' - R'(\gamma', J')\gamma' - R'(\gamma', J)\gamma'' \\ &\quad - R'(\gamma', J')\gamma' - R(\gamma'', J')\gamma' - R(\gamma', J'')\gamma' - R(\gamma', J')\gamma'' \\ &= -R''(\gamma', J)\gamma' - 2R'(\gamma', J')\gamma' - R(\gamma', J'')\gamma' \\ J^{(4)}(0) &= -R''(v, 0)v - 2R'(v, w)v - R(v, 0)v \\ &= -2R'(v, w)v \\ &= -2\nabla_v R(v, w)v \end{aligned}$$

Continuing along, we eventually get to this table:

$$\begin{aligned} J(0) &= 0 \\ J'(0) &= w \\ J''(0) &= 0 \\ J^{(3)}(0) &= -R(v, w)v \\ J^{(4)}(0) &= -2\nabla_v R(v, w)v \\ J^{(5)}(0) &= -3\nabla_v \nabla_v R(v, w)v + R(v, R(v, w)v)v \end{aligned}$$

And then plugging this into the expressions for $f^{(k)}$ gives:

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f''(0) &= 2\langle w, w\rangle \\ f^{(3)}(0) &= 0 \\ f^{(4)}(0) &= -8\langle R(v, w)v, w\rangle \\ f^{(5)}(0) &= -20\langle (\nabla_v R)(v, w)v, w\rangle \\ f^{(6)}(0) &= -36\langle (\nabla_v \nabla_v R)(v, w)v, w\rangle + 32\langle R(v, w)v, R(v, w)v\rangle. \end{aligned}$$

So, using this, along with the formula for the Taylor series centered at 0, we have that $f(t)$ can be written as:

$$f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f^{(3)}(0)}{3!}t^3 + \frac{f^{(4)}(0)}{4!}t^4 + \frac{f^{(5)}(0)}{5!}t^5 + \frac{f^{(6)}(0)}{6!}t^6 + O(t^6)$$

So therefore, $f(t) = \langle J(t), J(t)\rangle = |J(t)|^2$ has the expansion:

$$\begin{aligned} |J(t)|^2 =& \langle w, w\rangle t^2 - \frac{1}{3}R(v, w, v, w)t^4 - \frac{1}{6}\langle(\nabla_v R)(v, w)v, w\rangle t^5 \\ &+ \left[\frac{2}{45}\langle R(v, w)v, R(v, w)v\rangle - \frac{1}{20}\langle(\nabla_v\nabla_v R)(v, w)v, w\rangle\right] t^6 + o(t^6) \end{aligned}$$

From this expansion, as well as the fact that $J(t) = (d\exp_p)_{tv}(tw)$, we can repeat this same process as above to calculate $\langle(d\exp_p)_{tv}(tu), (d\exp_p)_{tv}(tw)\rangle$

This is done by considering two Jacobi fields $J_1(t)$ and $J_2(t)$ along the geodesic $\gamma(t) = \exp_p(tv)$ such that $\frac{DJ_1}{dt}(0) = u$ and $\frac{DJ_2}{dt}(0) = w$. Then, we can compute the inner product of these two Jacobi fields and expand it as a power series, deriving it in an identical way as above.

$$\langle J_1(t), J_2(t)\rangle = \langle(d\exp_p)_{tv}(tu), (d\exp_p)_{tv}(tw)\rangle =$$

$$\begin{aligned} &\langle u, w\rangle - \frac{1}{3}R(u, v, u, w)t^2 - \frac{1}{6}\langle(\nabla_v R)(u, v, u, w), v\rangle t^3 \\ &+ \left[\frac{2}{45}\langle R(u, v)u, R(v, w)v\rangle - \frac{1}{20}\langle(\nabla_v\nabla_v R)(u, v, u, w), v\rangle\right] t^4 + O(t^5) \end{aligned}$$

Now, if we let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM, then we can plug this in to the equation, along with $t = 1$ and get:

$$\langle(d\exp_p)_v(e_i), (d\exp_p)_v(e_j)\rangle =$$

$$\begin{aligned} &\langle e_i, e_j\rangle - \frac{1}{3}R(v, e_i, v, e_j) - \frac{1}{6}\langle(\nabla_v R)(v, e_i, v, e_j), v\rangle \\ &+ \left[\frac{2}{45}\langle R(v, e_i)v, R(v, e_j)v\rangle - \frac{1}{20}\langle(\nabla_v\nabla_v R)(v, e_i, v, e_j), v\rangle\right] + O(|v|^5) \end{aligned}$$

Now, also suppose that $B_\epsilon(p)$ is a geodesic ball with center p and radius $\epsilon > 0$, such that:

$$q = \exp_p\left(\sum_{k=1}^{n} x_k e_k\right) \in B_\epsilon(p)$$

Where $(x_1, \ldots, x_n)$ are the normal coordinates determined by $(e_1, \ldots, e_n)$. In this case, we have the relationship that:

$$\left.\frac{\partial}{\partial x_i}\right|_q = (d\exp_p)_{\sum_{k=1}^n x_k e_k}(e_i)$$

Which makes:

$$g_{ij}(x_1, \ldots, x_n) = \left\langle \left.\frac{\partial}{\partial x_i}\right|_q, \left.\frac{\partial}{\partial x_j}\right|_q \right\rangle = \langle(d\exp_p)_{\sum_{k=1}^n x_k e_k}(e_i), (d\exp_p)_{\sum_{k=1}^n x_k e_k}(e_j)\rangle$$

So then, on $B_\epsilon(p)$, we have that:

$$\nabla R = \sum_{i,j,k,l,m} R_{ijkl,m} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \otimes dx^m$$
$$\nabla\nabla R = \sum_{i,j,k,l,r,s} R_{ijkl,rs} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \otimes dx^r \otimes dx^s$$

So that when we substitute in the necessary components, we end up with:

$$\begin{aligned} g_{ij}(x) =& \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{ikjl}(p)x^k x^l - \frac{1}{6}\sum_{k,l,m} R_{ijkl,m}(p)x^k x^l x^m \\ &- \frac{1}{20}\sum_{k,l,r,s} R_{ikjl,rs}(p)x^k x^l x^r x^s + \frac{2}{45}\sum_{k,l,r,s,m} R_{iklm}(p)R_{jrsm}(p)x^k x^l x^r x^s + O(|x|^5) \end{aligned}$$

Which is a taylor expansion of the metric tensor.

2.4 Taylor Expansion of $\sqrt{\det(g_{ij})}$

Now, if we let $g(x) = (g_{ij}(x))$, then we can see:

$$g(x) = I + g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) + O(|x|^5)$$

Where I is the identity matrix, and $g^{(k)}(x)$ is the kth order term in the taylor expansion of $g(x)$.

Before continuing, we need to establish the following identity:

Lemma 2.4. *Let A be any positive definite $n \times n$ matrix with n eigenvalues, $\{\lambda_i\}_{i=1}^n$, such that $\log(A)$ is well defined. Then,*

$$\sqrt{\det A} = \exp\left(\frac{1}{2}\operatorname{Tr}(\log(A))\right)$$

Proof:

First, let us consider the eigenvalue decomposition of A. That is, we can write A as:

$$A = Q\Lambda Q^{-1}$$

Where Q is an orthogonal matrix, and Λ is a diagonal matrix with the eigenvalues of A on the diagonal. Then, we can see that:

$$\det(A) = \det(Q)\det(\Lambda)\det(Q^{-1}) = \det(\Lambda) = \prod_{i=1}^{n} \lambda_i$$

Which makes $\log(\det(A)) = \sum_{i=1}^n \log(\lambda_i)$

Now, we can also consider a matrix $\log(A)$ which is an $n \times n$ matrix as well, with eigenvalues $\{\log(\lambda_i)\}_{i=1}^n$. Remember as well that the trace of a matrix can be calculated as the sum of the eigenvalues, so:

$$\mathrm{Tr}(\log(A)) = \sum_{i=1}^{n} \log(\lambda_i)$$

We can see that both $\log(\det(A))$ and $\mathrm{Tr}(\log(A))$ are equal, so:

$$\sqrt{\det(A)} = \sqrt{\exp(\log(\det(A)))} = \exp\left(\frac{1}{2}\log(\det(A))\right) = \exp\left(\frac{1}{2}\mathrm{Tr}(\log(A))\right)$$

So we have proved the lemma that:

$$\sqrt{\det(A)} = \exp\left(\frac{1}{2}\mathrm{Tr}(\log(A))\right)$$

Now that we have proven this, we can move forward with the calculation.

We know that $\log(I + A)$, given necessary restrictions, can be expanded as a taylor series as:

$$\log(I + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + O(|A|^5)$$

Substituting the equation $g(x) = I + g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) + O(|x|^5)$ into this, we can see that $g(x) = I + A$, where $A = g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) + O(|x|^5)$. Therefore,

$$\begin{aligned}\log(g(x)) = &\left(g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x)\right) - \frac{\left(g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x)\right)^2}{2} \\ &+ \frac{\left(g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x)\right)^3}{3} - \frac{\left(g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x)\right)^4}{4} + O(|x|^5)\end{aligned}$$

But since we are only expanding up to the 5$^{\text{th}}$ order, we can ignore many of these terms, such as $g^{(3)}(x)^2$, $g^{(4)}(x)^2$, and everything after that. So we can simplify this expression greatly, and come to the equation:

$$\log(g(x)) = g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) - \frac{g^{(2)}(x)^2}{2} + O(|x|^5)$$

Using the expansion that was already derived for $g(x)$, we see that:

$$-\frac{1}{2}g^{(2)}(x)^2 = -\frac{1}{18}\sum_{k,l,r,s,m} R_{iklm}R_{jrsm}x^k x^l x^r x^s$$

Giving the final answer for $\log(g(x))$ as:

$$\begin{aligned}\log(g(x))_{ij} = &-\frac{1}{3}\sum_{k,l} R_{ikjl}(p)x^k x^l - \frac{1}{6}\sum_{k,l,m} R_{ijkl,m}(p)x^k x^l x^m \\ &- \frac{1}{20}\sum_{k,l,r,s} R_{ikjl,rs}(p)x^k x^l x^r x^s + \frac{2}{45}\sum_{k,l,r,s,m} R_{iklm}(p)R_{jrsm}(p)x^k x^l x^r x^s \\ &- \frac{1}{18}\sum_{k,l,r,s,m} R_{iklm}R_{jrsm}x^k x^l x^r x^s + O(|x|^5)\end{aligned}$$

Taking the trace of this object involves simply summing over all values where $i = j$, i.e. $\mathrm{Tr}(\log(g(x))) = g^{ij}\log(g(x))_{ij}$. So we can see that:

$$\begin{aligned}\log(g(x))_{ij} =& -\frac{1}{3}\sum_{k,l} g^{ij}R_{ikjl}(p)x^k x^l - \frac{1}{6}\sum_{k,l,m} g^{ij}R_{ijkl,m}(p)x^k x^l x^m \\ &- \frac{1}{20}\sum_{k,l,r,s} g^{ij}R_{ikjl,rs}(p)x^k x^l x^r x^s - \frac{1}{90}\sum_{k,l,r,s,m} g^{ij}R_{iklm}(p)R_{jrsm}(p)x^k x^l x^r x^s \\ &+ O(|x|^5) \\ &= -\frac{1}{3}\sum_{k,l} R_{kl}(p)x^k x^l - \frac{1}{6}\sum_{k,l,m} R_{kl,m}(p)x^k x^l x^m - \frac{1}{20}\sum_{k,l,r,s} R_{kl,rs}(p)x^k x^l x^r x^s \\ &- \frac{1}{90}\sum_{i,k,l,r,s,m} R_{iklm}(p)R_{irsm}(p)x^k x^l x^r x^s\end{aligned}$$

Then, we have our result by the fact that $\sqrt{\det(g(x))} = \exp\left(\frac{1}{2}\mathrm{Tr}(\log(g(x)))\right)$

$$\begin{aligned}&\sqrt{\det(g(x))} = \\ &\quad 1 - \frac{1}{6}\sum_{k,l} R_{kl}(p)x^k x^l - \frac{1}{12}\sum_{k,l,m} R_{k,l,m}(p)x^k x^l x^m \\ &\quad - \sum_{k,l,r,s}\left(-\frac{1}{40}\sum_{k,l,r,s} R_{kl,rs}(p) - \frac{1}{180}\sum_{i,m} R_{iklm}(p)R_{irsm}(p) + \frac{1}{72}R_{kl}(p)R_{rs}(p)\right)x^k x^l x^r x^s \\ &\quad + O(|x|^5)\end{aligned}$$

3 Isometric Immersions

Let (M,g) and $(\widetilde{M},\widetilde{g})$ be Riemannian manifolds, and $f: M \to \widetilde{M}$ be a differentiable immersion of a manifold M of dimension n into a manifold $\widetilde{M}$ of dimension $\widetilde{n}$. The Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ induces a Riemannian metric g on M.

Definition 3.1. We call $f: M \to \widetilde{M}$ an isometric immersion if for any $v_1, v_2 \in T_pM$

$$g(v_1, v_2) = \widetilde{g}(df_p(v_1), df_p(v_2))$$

For the rest of the section, let ∇ be the Levi-Civita connection on (M,g), and $\widetilde{\nabla}$ be the Levi-Civita connection on $(\widetilde{M},\widetilde{g})$. Also, $D := f^*\widetilde{\nabla}$ is the pullback connection on $f^*T\widetilde{M}$.

3.1 Normal Bundle

For any $p \in M$, $T_{f(p)}\widetilde{M} = (f^*TM)_p = T_pM \oplus \left(T_pM\right)^{\perp}$. This is known as the orthogonal decomposition of $T_{f(p)}\widetilde{M}$.

We want to identify T_pM with $df_p\left(T_pM\right) \subset T_{f(p)}\widetilde{M}$

Notation: For any $v \in T_{f(p)}\widetilde{M} = (f^*TM)_p$, we will write $v = v^T + v^N$, where $v^T \in T_pM$ and $v^N \in \left(T_pM\right)^N$.

Definition 3.2. We define the *normal bundle* of the isometric immersion $f: (M,g) \to (\widetilde{M},\widetilde{g})$ to be

$$N(f) = \bigcup_{p \in M} \left(T_pM\right)^{\perp} \subset \bigcup_{p \in M} \left(f^*TM\right)_p = f^*TM$$

It is a rank $k = \widetilde{n} - n$ C^∞ vector bundle over M. It is also a rank k C^∞ subbundle of $f^*T\widetilde{M}$, for which we can see the orthogonal composition of below.

$$f^*T\widetilde{M} = TM \oplus N(f)$$

$$C^\infty(M, f^*T\widetilde{M}) = C^\infty(M, TM) \oplus C^\infty(M, N(f))$$

In do Carmo's [dC] notation, $C^\infty(M, TM) = \mathfrak{X}(M)$, and $C^\infty(M, N(f)) = \mathfrak{X}(M)^{\perp}$.

Notation:

If $f: M \to \widetilde{M}$ is an immersion, we have that by definition, $\forall p \in M$, the following maps are injective, as an $\mathbb{R}$-linear map between vector spaces, or as a morphism between $C^\infty(M)$-modules:

$$df_p : T_pM \to T_{f(p)}\widetilde{M}$$
$$f_* : \mathfrak{X}(M) \to C^\infty(M, f^*T\widetilde{M})$$

We sometimes identify $X \in \mathfrak{X}(M)$ with $f_*(X) \in C^\infty(M, f^*T\widetilde{M})$.

For $X, Y \in \mathfrak{X}(M) \subset C^\infty(M, f^*T\widetilde{M})$, we have that $D_X(f_*Y) \in C^\infty(M, f^*T\widetilde{M})$

Note that we can decompose $D_X(f_*Y) = \left(D_X\left(f_*Y\right)\right)^T + \left(D_X\left(f_*Y\right)\right)^N$

Also, it is possible to prove $\left(D_X\left(f_*Y\right)\right)^T = f_*(\nabla_X Y) \in C^\infty(M, f^*T\widetilde{M})$, and $(D_XY)^T = \nabla_X Y$

Definition 3.3. Let $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion, and $D := f^*\widetilde{\nabla}$

Define a map $B(X, Y)$ by:

$$\begin{aligned} B : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)^{\perp} \\ (X, Y) &\mapsto \big(D_X(f_*Y)\big)^N \end{aligned}$$

Lemma 3.1. *For any $X, Y \in \mathfrak{X}(M)$:*

(i) $B(X, Y)$ is a $C^\infty(M)$-bilinear map

(ii) B is symmetric

(iii) $B \in C^\infty(M, \mathrm{Sym}^2 T^*M \otimes N(f))$

Proof:

Note that $B(X, Y)$ can also be defined in another way. Let $X, Y \in \mathfrak{X}(M)$, and $\widetilde{X}, \widetilde{Y}$ be extensions of X, Y to $\widetilde{M}$. We see that $\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\right)^T$, which means an alternate definition of B is:

$$B(X, Y) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \nabla_X Y$$

We can see that $B(X, Y)$ is $C^\infty(M)$-linear by the following argument:

$$B(fX, Y) = \widetilde{\nabla}_{f\widetilde{X}}\widetilde{Y} - \nabla_{fX} Y = f\left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \nabla_X Y\right) = fB(X, Y)$$

$$B(X, fY) = \widetilde{\nabla}_{\widetilde{X}} f\widetilde{Y} - \nabla_X fY = \widetilde{f}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - f\nabla_X Y + \widetilde{X}(\widetilde{f})\widetilde{Y} - X(f)Y = f\left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \nabla_X Y\right) = fB(X, Y)$$

Additionally, to see that $B(X, Y)$ is symmetric:

$$B(X, Y) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \nabla_X Y = \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} + [\widetilde{X}, \widetilde{Y}] - \nabla_Y X - [X, Y] = \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - \nabla_Y X = B(Y, X)$$

So we have shown that $B(X, Y)$ is a symmetric bilinear function.

3.2 Second Fundamental Form and Shape Operator

Definition 3.4. The *second fundamental form* of the isometric immersion $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ at $p \in M$ along $\eta \in (T_pM)^{\perp}$ is defined to be:

$$\begin{aligned} H_\eta : T_pM \times T_pM &\to \mathbb{R} \\ H_\eta(x, y) &\mapsto \big\langle B(x, y), \eta \big\rangle \end{aligned}$$

This gives rise to another mapping (sometimes also called the second fundamental form).

$$\mathrm{I\!I}_\eta(x) = H_\eta(x, x)$$

Definition 3.5. Define the shape operator as a mapping:

$$S_\eta : T_pM \to T_pM$$

such that for any $x, y \in T_pM$, we have that:

$$\langle S_\eta(x), y\rangle = H_\eta(x, y)$$

Proposition 3.6. The shape operator is self adjoint, i.e., $\langle S_\eta(x), y\rangle = \langle x, S_\eta(y)\rangle$

Proof:

$$\begin{aligned}
\langle S_\eta(x), y\rangle &= H_\eta(x, y) \\
&= \big\langle B(x, y), \eta\big\rangle \\
&= \big\langle B(y, x), \eta\big\rangle \\
&= H_\eta(y, x) \\
&= \langle S_\eta(y), x\rangle \\
&= \langle x, S_\eta(y)\rangle
\end{aligned}$$

Lemma 3.2. *Let $X \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(M)^\perp$. Then $S_\eta(X) = -\left(D_X\eta\right)^\perp$*

Proof:

$$\begin{aligned}
\langle S_\eta(X), Y\rangle &= \big\langle B(X, Y), \eta\big\rangle \\
&= \langle \big(D_X(Y)\big)^N, \eta\rangle \\
&= \langle \big(D_X(Y)\big)^N, \eta\rangle + 0 \\
&= \langle \big(D_X(Y)\big)^N, \eta\rangle + \langle \big(D_X(Y)\big)^T, \eta\rangle \\
&= \langle D_X(Y), \eta\rangle \\
&= \langle D_X(Y), \eta\rangle + \langle Y, D_X\eta\rangle - \langle Y, D_X\eta\rangle \\
&= X\langle Y, \eta\rangle - \langle Y, D_X\eta\rangle \\
&= -\langle Y, D_X\eta\rangle \quad (Y \in \mathfrak{X}(M) \text{ and } \eta \in \mathfrak{X}(M)^\perp \implies \langle Y, \eta\rangle = 0) \\
&= \langle -(D_X\eta)^T, Y\rangle \quad (\text{because the } (D_X\eta)^\perp \text{ term vanishes in the inner product})
\end{aligned}$$

So we have shown that $S_\eta(X) = -(D_X\eta)^\perp$

□

Corollary 3.7. An immediate corollary of Lemma 3.2 is that if $\dim(\widetilde{M}) = \dim(M) + 1$, then we have the existence of a unit normal $\eta \in C^\infty(M, N(f)), \langle \eta, \eta\rangle = 1$, which implies $S_\eta(X) = -D_X\eta \quad \forall x \in \mathfrak{X}(M)$.

η exists $\iff$ $N(f)$ is trivial.

Additionally, η is unique up to sign if M is connected.

We can see that this is true because:

$$
\begin{aligned}
(D_x\eta)^N &= \langle D_X\eta, \eta\rangle\eta \\
&= \left(\frac{1}{2}X\langle\eta,\eta\rangle\right)\eta \\
&= \frac{1}{2}(0)\eta \\
&= 0
\end{aligned}
$$

So we do not need to worry about the normal component of the derivative, meaning in this case,

$$S_\eta(x) = -(D_X\eta)^T = -(D_X\eta)^T - (D_X\eta)^N + (D_X\eta)^N = -(D_X\eta) + (0) = -D_X\eta$$

Example 3.1. Calculate the second fundamental form of $f : (S^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$ along the inward unit normal.

On the sphere, $\forall p \in S^n \quad \eta(p) = -p$.

Let $\widetilde{\eta} = \sum_{j=1}^{n+1} X_j \frac{\partial}{\partial x_j} \in \mathfrak{X}(\mathbb{R}^{n+1})$

Then for any $p \in S^n$, we have $\widetilde{\eta}(p) = \eta(p)$

Consider the covariant derivative $\widetilde{\nabla}$ defined by g_0 on $\mathbb{R}^{n+1}$, so that:

$$\widetilde{\nabla}\widetilde{\eta} = -\sum_{j=1}^{n+1} dx_j \otimes \frac{\partial}{\partial x_j} \in C^\infty(\mathbb{R}^{n+1}, \mathrm{End}(T\mathbb{R}^{n+1}))$$

For all $p \in \mathbb{R}^{n+1}$, we have:

$$
\begin{aligned}
\left(\widetilde{\nabla}\widetilde{\eta}\right)_p : T_p\mathbb{R}^{n+1} &\to T_p\mathbb{R}^{n+1} \\
\frac{\partial}{\partial x_j}(p) &\mapsto -\frac{\partial}{\partial x_j}(p)
\end{aligned}
$$

Which means that $\left(\widetilde{\nabla}\widetilde{\eta}\right)_p = -\,\mathrm{id}\, T_p\mathbb{R}^{n+1}$, or equivalently, $\widetilde{\nabla}_v\widetilde{\eta} = -v \quad \forall v \in T_p\mathbb{R}^{n+1}$

Recall that $D = f^*\widetilde{\nabla}$ and $\eta = f^*\widetilde{\eta}$ so that for all $p \in S^n$ and $v \in T_pS^n$

$$S_\eta(v) = -D_v\eta = -\widetilde{\nabla}_v\widetilde{\eta} = v$$

$$H_\eta(X,Y) = \langle S_\eta(X), Y\rangle = \langle X, Y\rangle$$

Which immediately lets us conclude that $H_\eta = g_{can}^{S^n}$

$\square$

3.3 Connections on the Normal Bundle

We define a connection $\nabla^{\perp}$ on $N(f)$ by:

$$\nabla^{\perp}_X \eta = (D_X \eta)^N$$

Choose $X, Y \in \mathfrak{X}(M)$, and $\eta \in \mathfrak{X}(M)^{\perp}$. Then we have the following:

$$\begin{aligned} D_X Y &= (D_X Y)^T + (D_X Y)^N = \nabla_X Y + B(X, Y) \\ D_X \eta &= (D_X \eta)^T + (D_X \eta)^N = -S_\eta(X) + \nabla^{\perp}_X \eta \end{aligned}$$

where:

$$\begin{aligned} \nabla &\text{ is a connection on } TM, T^*M, (TM)^{\otimes r} \otimes (T^*M)^{\otimes s} \\ \nabla^{\perp} &\text{ is a connection on } N(f), N(f)^*, N(f)^{\otimes \ell} \otimes \left(N(f)^*\right)^{\otimes m} \end{aligned}$$

$(N(f))^*$ is called the *conormal bundle.*

These connections allow us to define a general connection on $(TM)^{\otimes r} \otimes (T^*M)^{\otimes s} \otimes N(f)^{\otimes \ell} \otimes \left(N(f)^*\right)^{\otimes m}$

In particular, let us define $B \in C^\infty \left(M, (T^*M)^{\otimes^2} \otimes N(f)^*\right)$ by:

$$B(Y, Z, \eta) = \langle B(Y, Z), \eta \rangle$$

with $Y, Z \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(M)^{\perp}$

So that for $X \in \mathfrak{X}(M)$, we can define:

$$D_X B \in C^\infty(M, (T^*M)^{\otimes^2} \otimes N(f)^*)$$

as

$$(D_X B)(Y, Z, \eta) = X(B(Y, Z, \eta)) - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta) + B(Y, Z, \nabla^{\perp}_X \eta)$$

with $X, Y, Z \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(M)^{\perp}$

3.4 Normal Curvature

Let us first define some terms related to curvature. These will be necessary for defining the Guass, Codazzi, and Ricci equations for isometric immersions.

Recall that on a Riemannian manifold (M, g), the curvature tensor R is defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

We can generalize this in the following way:

$$
\begin{aligned}
&R_{\overline{\nabla}} \in \Omega^2\left(\widetilde{M}, \mathrm{End}\left(T\widetilde{M}\right)\right)\\
&R := f^* R_{\overline{\nabla}} = R_{f^*\overline{\nabla}} = R_D \in \Omega^2\left(M, \mathrm{End}\left(f^* T\widetilde{M}\right)\right)\\
&R = R_{\nabla} \in \Omega^2\left(M, \mathrm{End}\left(TM\right)\right)\\
&R^{\perp} = R_{\nabla^{\perp}} \in \Omega^2\left(M, \mathrm{End}\left(N(f)\right)\right)
\end{aligned}
$$

Note that $\mathrm{End}(f^*T\widetilde{M})$ is the space of all automorphisms of $f^*T\widetilde{M}$, or $C^\infty(M)$-linear maps from $f^*T\widetilde{M}$ to itself. We can visualize some of the equations that we will be deriving in the following picture:

Gauss	Codazzi
Codazzi	Ricci

3.4.1 Gauss Equation

Proposition 3.8. The Gauss Equation is given by:

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle + \langle B(X,T),B(Y,Z)\rangle - \langle B(X,Z),B(Y,T)\rangle$$

Proof:

First recall that $\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)$

So then we have:

$$\begin{aligned}
\overline{R}(X,Y)Z &= \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_X\overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]}Z \\
&= \overline{\nabla}_Y(\nabla_X Z + B(X,Z)) - \overline{\nabla}_X(\nabla_Y Z + B(Y,Z)) + \nabla_{[X,Y]}Z + B([X,Y],Z) \\
&= \nabla_Y\nabla_X Z + \nabla_Y B(X,Z) + B(Y,\nabla_X Z + B(X,Z)) \\
&- \nabla_X\nabla_Y Z - \nabla_X B(Y,Z) - B(X,\nabla_Y Z + B(Y,Z)) \\
&+ \nabla_{[X,Y]}Z + B([X,Y],Z) \\
&= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z + \nabla_Y B(X,Z) - \nabla_X B(Y,Z) \\
&+ B([X,Y],Z) + B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B(Y,B(X,Z)) - B(X,B(Y,Z)) \\
&= R(X,Y)Z + \nabla^{\perp}_Y B(X,Z) - \nabla^{\perp}_X B(Y,Z) + B([X,Y],Z) + B(Y,\nabla_X Z) - B(X,\nabla_Y Z) \\
&+ B(Y,B(X,Z)) - B(X,B(Y,Z)) \\
&= R(X,Y)Z + \nabla^{\perp}_Y B(X,Z) - \nabla^{\perp}_X B(Y,Z) + B([X,Y],Z) + B(Y,\nabla_X Z) - B(X,\nabla_Y Z) \\
&- S_{B(X,Z)}(Y) + S_{B(Y,Z)}(X)
\end{aligned}$$

And now taking this inner product with T, we have:

$$\begin{aligned}
\langle \overline{R}(X,Y)Z,T\rangle &= \langle R(X,Y)Z,T\rangle + \langle \nabla^{\perp}_Y B(X,Z),T\rangle - \langle \nabla^{\perp}_X B(Y,Z),T\rangle \\
&+ \langle B([X,Y],Z),T\rangle + \langle B(Y,\nabla_X Z),T\rangle - \langle B(X,\nabla_Y Z),T\rangle \\
&- \langle S_{B(X,Z)}(Y),T\rangle + \langle S_{B(Y,Z)}(X),T\rangle \\
&= \langle R(X,Y)Z,T\rangle + \langle B(X,T),B(Y,Z)\rangle - \langle B(X,Z),B(Y,T)\rangle
\end{aligned}$$

Notice that we took advantage of two very important formulas:

1. $B(X,T) = -S_T(X)$
2. $\langle S_\eta(X),Y\rangle = \langle B(X,Y),\eta\rangle$

As well as the fact that $\nabla^{\perp}$ is a connection on $N(f)$, so that the inner product of $\nabla^{\perp}$ with any tangent vector T vanishes. Therefore, we have the desired result:

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle + \langle B(X,T),B(Y,Z)\rangle - \langle B(X,Z),B(Y,T)\rangle$$

□

3.4.2 Codazzi Equation

Proposition 3.9. The Codazzi Equation is given by:

$$\langle \overline{R}(X,Y)Z,\eta\rangle = \langle \overline{R}(X,Y)\eta, Z\rangle = (D_Y B)(X,Z,\eta) - (D_X B)(Y,Z,\eta)$$

Proof:

Given an isometric immersion, let us denote the space of vector fields normal to M by $\mathfrak{X}(M)^{\perp}$. The second fundamental form of the immersion can then be thought of as a tensor:

$$B : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} \to \mathbb{R}$$

which is defined by $B(X,Y,\eta) = \langle B(X,Y),\eta\rangle$

And this allows us to naturally extend the definition of the covariant derivative as:

$$\left(\overline{\nabla}_X B\right)(Y,Z,\eta) = X(B(Y,Z,\eta)) - B(\nabla_X Y, Z,\eta) - B(Y,\nabla_X Z,\eta) + B(Y,Z,\nabla^{\perp}_X \eta)$$

Then using this notation, we see that:

$$\begin{aligned}\left(\overline{\nabla}_X B\right)(Y,Z,\eta) &= X(B(Y,Z,\eta)) - B(\nabla_X Y, Z,\eta) - B(Y,\nabla_X Z,\eta) + B(Y,Z,\nabla^{\perp}_X \eta)\\ &= \langle \nabla^{\perp}_X(B(Y,Z)),\eta\rangle - \langle B(\nabla_X Y,Z),\eta\rangle - \langle B(Y,\nabla_X Z),\eta\rangle\end{aligned}$$

Recall from the proof of the Guass equation that we have:

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ B(Y,\nabla_X Z) + \nabla^{\perp}_Y B(X,Z) - S_{B(X,Z)}Y\\ &- B(X,\nabla_Y Z) - \nabla^{\perp}_X B(Y,Z) + S_{B(Y,Z)}X + B([X,Y],Z).\end{aligned}$$

Now using this, we can immediately see:

$$\begin{aligned}\langle \overline{R}(X,Y)Z,\eta\rangle &= \langle R(X,Y)Z,\eta\rangle + \langle B(Y,\nabla_X Z),\eta\rangle + \langle \nabla^{\perp}_Y B(X,Z),\eta\rangle - \langle S_{B(X,Z)}Y,\eta\rangle\\ &- \langle B(X,\nabla_Y Z),\eta\rangle - \langle \nabla^{\perp}_X B(Y,Z),\eta\rangle + \langle S_{B(Y,Z)}X,\eta\rangle + \langle B([X,Y],Z),\eta\rangle\\ &= \langle B(Y,\nabla_X Z),\eta\rangle + \langle \nabla^{\perp}_Y B(X,Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle\\ &- \langle \nabla^{\perp}_X B(Y,Z),\eta\rangle + \langle B(\nabla_X Y,Z),\eta\rangle - \langle B(\nabla_Y X,Z)\eta\rangle\end{aligned}$$

Then from this, notice that:

$$\begin{aligned}\langle \nabla^{\perp}_Y(B(X,Z)),\eta\rangle &= Y\langle B(X,Z),\eta\rangle - \langle B(X,Z), D_Y\eta\rangle\\ &= Y(B(X,Z,\eta)) - \langle B(X,Z),\nabla^{\perp}_Y\eta\rangle\\ &= (D_Y B)(X,Z,\eta) + B(\nabla_Y X,Z,\eta) + B(X,\nabla_Y Z,\eta)\end{aligned}$$

And this lets us conclude:

$$(D_Y B)(X, Z, \eta) = \langle \nabla_Y^\perp (B(X, Z)), \eta \rangle - B(\nabla_Y X, Z, \eta) - B(X, \nabla_Y Z, \eta)$$

Using this equivalence, we can greatly simplify the above expression for $\langle \overline{R}(X, Y)Z, \eta \rangle$:

$$\langle \overline{R}(X, Y)Z, \eta \rangle = (D_Y B)(X, Z, \eta) - (D_X B)(Y, Z, \eta)$$

Which is the Codazzi equation.

□

3.4.3 Ricci Equation

Proposition 3.10. The Ricci Equation is given by:

$$\langle \overline{R}(X,Y)\eta, \xi\rangle = \langle R^{\perp}(X,Y)\eta, \xi\rangle + \langle \left[S_\eta, S_\xi\right] X, Y\rangle$$

where

$$\left[S_\eta, S_\xi\right] X = S_\eta \circ S_\xi(X) - S_\xi \circ S_\eta(X)$$

Proof:

First, recall that we have:

$$\overline{\nabla}_X \eta = \nabla^{\perp}_X \eta - S_\eta(X)$$

And then consider:

$$\begin{aligned}
\overline{R}(X,Y)\eta &= \overline{\nabla}_Y \overline{\nabla}_X \eta - \overline{\nabla}_X \overline{\nabla}_Y \eta + \overline{\nabla}_{[X,Y]}\eta \\
&= \overline{\nabla}_Y(\nabla^{\perp}_X \eta + B(X,\eta)) - \overline{\nabla}_X(\nabla^{\perp}_Y \eta + B(Y,\eta)) + \nabla^{\perp}_{[X,Y]}\eta + B([X,Y],\eta) \\
&= \overline{\nabla}_Y(\nabla^{\perp}_X \eta - S_\eta(X)) - \overline{\nabla}_X(\nabla^{\perp}_Y \eta - S_\eta(Y)) + \nabla^{\perp}_{[X,Y]}\eta - S_\eta([X,Y]) \\
&= \overline{\nabla}_Y \nabla^{\perp}_X \eta - \overline{\nabla}_Y S_\eta(X) - \overline{\nabla}_X \nabla^{\perp}_Y \eta + \overline{\nabla}_X S_\eta(Y) + \nabla^{\perp}_{[X,Y]}\eta - S_\eta([X,Y]) \\
&= \nabla^{\perp}_Y \nabla^{\perp}_X \eta - S_{\nabla^{\perp}_X \eta} Y - \nabla^{\perp}_Y S_\eta(X) + S_{S_\eta(X)} Y - \nabla^{\perp}_X \nabla^{\perp}_Y \eta + S_{\nabla^{\perp}_Y \eta} X \\
&+ \nabla^{\perp}_X S_\eta Y - S_{S_\eta Y} X + \nabla^{\perp}_{[X,Y]}\eta - S_\eta([X,Y])
\end{aligned}$$

Then using the fact that $S_\eta X = -B(X,\eta)$, we have:

$$\begin{aligned}
\overline{R}(X,Y)\eta &= \nabla^{\perp}_Y \nabla^{\perp}_X \eta - \nabla^{\perp}_X \nabla^{\perp}_Y \eta + \nabla^{\perp}_{[X,Y]}\eta - B(Y, S_\eta X) + B(X, S_\eta Y) \\
&- S_{\nabla^{\perp}_X \eta} Y + S_{\nabla^{\perp}_Y \eta} X + \nabla^{\perp}_X S_\eta Y - \nabla^{\perp}_Y S_\eta X - S_\eta([X,Y])
\end{aligned}$$

And then multiplying both sides by ξ, while remembering that $\langle B(X,Y),\eta\rangle = \langle S_\eta X, Y\rangle$, while also noticing that since ξ is a tangent vector and orthogonal to η, the terms involving S_η disappear, we obtain:

$$\begin{aligned}
\langle \overline{R}(X,Y)\eta, \xi\rangle &= \langle R^{\perp}(X,Y)\eta, \xi\rangle - \langle B(S_\eta X, Y), \xi\rangle + \langle B(S_\eta Y, X), \xi\rangle \\
&= \langle R^{\perp}(X,Y)\eta, \xi\rangle + \langle (S_\eta S_\xi - S_\xi S_\eta) X, Y\rangle \\
&= \langle R^{\perp}(X,Y)\eta, \xi\rangle + \langle \left[S_\eta, S_\xi\right] X, Y\rangle
\end{aligned}$$

So we have proven the Ricci equation which states:

$$\langle \overline{R}(X,Y)\eta, \xi\rangle = \langle R^{\perp}(X,Y)\eta, \xi\rangle + \langle \left[S_\eta, S_\xi\right] X, Y\rangle$$

□

3.5 Corollaries of the Gauss, Codazzi, and Ricci Equations

Let $p \in M$, and $x, y \in T_pM$ be orthonormal. Then we can define

$$\begin{aligned} \sigma &:= \mathbb{R}x \oplus \mathbb{R}y \subset T_pM \subset T_{f(p)}\widetilde{M} \\ K(x,y) &:= K(\sigma) = R(x,y,x,y) \\ \overline{K}(x,y) &:= \overline{K}(\sigma) = \overline{R}(x,y,x,y) \end{aligned}$$

Which means that the Guass equation in this case can be rewritten as:

$$\overline{K}(x,y) = K(x,y) - \langle B(x,x), B(y,y)\rangle + |B(x,y)|^2$$

Or equivalently:

$$K(x,y) - \overline{K}(x,y) = \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2$$

3.5.1 Example with S^n

In particular, if we have:

$$f : (M,g) = (S^n, g_{can}) \hookrightarrow (\widetilde{M}, \widetilde{g}) = (\mathbb{R}^{n+1}, g_0)$$

We already know that $\forall p \in S^n$, $\eta(p) = -p$, and $\forall x, y \in T_pS^n$, we have $B(x,y) = H_\eta(x,y)\eta = \langle x, y\rangle\eta$, when x and y are orthonormal.

$$\begin{aligned} K(x,y) - \overline{K}(x,y) &= \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2 \\ K(x,y) - 0 &= \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2 \\ K(x,y) &= 1 \cdot 1 - 0^2 \\ K(x,y) &= 1 \end{aligned}$$

So we have shown that (S^n, g_{can}) has constant sectional curvature equal to 1, for any $n \geq 2$.

4 Geodesic Manifolds

Definition 4.1. Let $f : (M, g) \to (\overline{M}, \overline{g})$ be an isometric immersion.

We say that f is *geodesic* at $p \in M$ if:

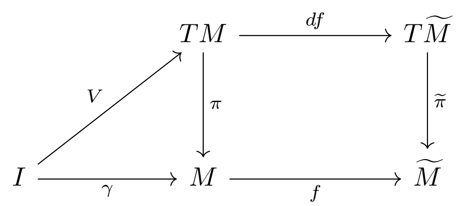
$$B(p) : T_pM \times T_pM \to (T_pM)^{\perp}$$

is zero. Alternatively, this is true if and only if $\forall \eta \in (T_pM)^{\perp}$, $H_\eta = 0$.

We say that f is totally geodesic if f is geodesic at every $p \in M$.

Lemma 4.1. *Let $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion, and I an open interval.*

Also consider the following commutative diagram



- *$\gamma : I \to M$ is a C^∞ curve in M*
- *$f \circ \gamma : I \to \widetilde{M}$ is a C^∞ curve in $\widetilde{M}$*
- *V is a C^∞ vector field along γ*
- *$\widetilde{V} := df \circ V : I \to T\widetilde{M}$ is a C^∞ vector field along $f \circ \gamma$*

Then we have:

$$\frac{\widetilde{D}}{dt}\widetilde{V}(t) = \frac{D}{dt}V(t) + B(\gamma'(t), V(t))$$

Where $\frac{\widetilde{D}}{dt}$ is defined by $(f \circ \gamma)^*\widetilde{\nabla} = \gamma^* D$, and $D = f^*\widetilde{\nabla}$.

Proof:

Both $\frac{\widetilde{D}}{dt}\widetilde{V}(t) - \frac{D}{dt}V(t)$ and $B(\gamma'(t), V(t))$ are $C^\infty(I)$-linear in $V(t)$.

So it suffices to check this when $V(t) = X(\gamma(t))$ and $X \in \mathfrak{X}(M)$, then:

$$\frac{\widetilde{D}}{dt}\widetilde{V}(t) = D'_\gamma(t)X$$
$$\frac{D}{dt}V(t) = \nabla_{\gamma'(t)}X$$

$$\begin{aligned}\frac{\widetilde{D}}{dt}\widetilde{V}(t) - \frac{D}{dt}V(t) &= D'_\gamma(t)X - \nabla_{\gamma'(t)}X \\ &= B(\gamma'(t), X(\gamma(t))) \\ &= B(\gamma'(t), V(t))\end{aligned}$$

□

Proposition 4.2. Let $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion. Then f is geodesic at $p \in M$ if and only if:

$\forall \gamma : (-\epsilon, \epsilon) \to M$ geodesic in (M, g) such that $\gamma(0) = p$, $\widetilde{\gamma} = f \circ \gamma : (-\epsilon, \epsilon) \to \widetilde{M}$ is geodesic at D.

Proof:

$(f \circ \gamma)'(t) = df_{\gamma}(t)(\gamma'(t))$

And then by lemma 4.1 we know:

$$\frac{\widetilde{D}}{dt}\widetilde{V}(t) = \frac{D}{dt}V(t) + B(\gamma'(t), V(t))$$

$(\implies)$

1. f is geodesic at $\gamma(0) = p \in M \implies B(\gamma'(0), \gamma'(0)) = 0$
2. $\gamma(t)$ is a geodesic $\implies \frac{D}{dt}\gamma'(t) = 0$

And then from these two statements along with the lemma, we immediately see:

$$\tfrac{\widetilde{D}}{dt}\widetilde{\gamma}'(0) = 0 \iff \widetilde{\gamma} = f \circ \gamma \text{ is a geodesic at } 0$$

$(\impliedby)$

$\forall x, y \in T_pM$, we have $B(p)(x, y) = 0$. Since B is symmetric and bilinear, it suffices to show that for any $v \in T_pM$, $B(p)(v, v) = 0$.

We know that $\exists \epsilon > 0$ such that $\gamma(t) = \exp_p(tv)$ is defined.

$\gamma : (-\epsilon, \epsilon) \to M$ is a geodesic in (M, g), with $\gamma(0) = p$ and $\gamma'(0) = v$.

This implies $\frac{\widetilde{D}}{dt}\widetilde{\gamma}'(0) = 0$, so that:

$$\frac{\widetilde{D}}{dt}\widetilde{\gamma}'(0) = \frac{D}{dt}\gamma'(0) + B(p)(\gamma'(0), \gamma'(0)) = \frac{D}{dt}\gamma'(0) + B(p)(v, v) = 0 \implies B(p)(v, v) = 0$$

□

If f is totally geodesic, then $B(\gamma'(t), \gamma'(t)) = 0$ which implies that:

- $\widetilde{\gamma}$ is a geodesic in $(\widetilde{M}, \widetilde{g})$
- $\widetilde{\gamma} = \widetilde{\exp}_f(p)(t\, df_p(v))$
- $\widetilde{\gamma} = f \circ \exp_p(tv)$

And we can see that:

$$f \circ \exp_p(tv) = \widetilde{\exp}_f(p) \circ df_p : B_\epsilon(0) \to \widetilde{M}$$

$$\begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}\widetilde{M} \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \widetilde{\exp}_{f(p)}} \\ M & \xrightarrow{f} & \widetilde{M} \end{array}$$

4.1 Examples of Totally Geodesic Isometric Embeddings

$$(\mathbb{R}^n, dx_1^2 + \cdots + dx_n^2) \hookrightarrow (\mathbb{R}^{n+k}, dx_1^2 + \cdots + dx_{n+k}^2)$$
$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$$

$$(S^{n-1}, g_{can}) \hookrightarrow (\mathbb{R}^{n+k-1}, g_{can})$$
$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$$
$$x_1^2 + \cdots + x_n^2 = 1$$

Let (M, g) be a Riemannian manifold, and $p \in M$.

$$\exists \epsilon > 0 \quad \exp_p : B_\epsilon(0) \to B_\epsilon(p)$$

Is a geodesic ball centered at p with radius $\epsilon > 0$, and $B_\epsilon(0) \subset T_pM$, $B_\epsilon(p) \subset M$.

Now let $\sigma \subset T_pM$ be a 2-plane.

$S = \exp_p(\sigma \cap B_\epsilon(0))$ is a 2 dimensional Riemannian submanifold of (M, g). Additionally, S is geodesic at p. Notice as well that:

$$K(p, \sigma) = K_S(p)$$

where $K(p, \sigma)$ is the sectional curvature of (M, g) and $K_S \in C^\infty(S)$ is the scalar curvature of S.

5 Curvature

5.1 Mean Curvature

Let $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion, $p \in M$, and $\eta \in (T_pM)^\perp$ such that $\langle \eta, \eta \rangle = 1$

Then the *mean curvature* of f at p along η is given by:

$$h_\eta := \frac{1}{n} \operatorname{Tr}(S_\eta)$$

where $n = \dim M$, and S_η is the shape operator of f at p along η.

Remember that $S_\eta : T_pM \to T_pM$ is self-adjoint.

Also let $e_1, \dots, e_n$ be an orthonormal basis of T_pM. Then we have:

$$S_\eta(e_i) = \sum_j A_{ij} e_j$$
$$A_{ij} = \langle S_\eta(e_i), e_j \rangle = \langle e_i, S_\eta(e_j) \rangle = A_{ji}$$

From this definition, we know that $\exists U \in O(n)$ such that:

$$A = U^{-1} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U = U^T \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U \qquad \lambda_1, \dots, \lambda_n \in \mathbb{R}$$

Then there exists an orthonormal basis $\widetilde{e}_1, \dots, \widetilde{e}_n$ of T_pM such that:

$$S_\eta(\widetilde{e}_i) = \lambda_i \widetilde{e}_i$$
$$\operatorname{Tr}(S_\eta) = \sum_{i=1}^n A_{ii} = \sum_{i=1}^n \lambda_i$$

Then from here, choose an orthonormal basis $E_1, \dots, E_k$ of $(T_pM)^\perp$ where $k = \dim \widetilde{M} - \dim M$

Then the *mean curvature vector* of f at p is given by:

$$\vec{H}(p) := \sum_{\alpha=1}^k h_{E_\alpha} E_\alpha = \frac{1}{n} \sum_{\alpha=1}^k \sum_{i=1}^n \langle B(e_i, e_i), E_\alpha \rangle = \frac{1}{n} \sum_{i=1}^n B(e_i, e_i) \in (T_pM)^\perp$$

Notice that this expression is independent of choice of basis for $\{E_\alpha\}_{\alpha=1}^k$ and $\{e_i\}_{i=1}^n$.

So $\vec{H} \in (T_pM)^\perp$ is the *mean curvature vector* of the isometric immersion $f : (M, g) \to (\widetilde{M}, \widetilde{g})$

We say that the isometric immersion f is *minimal* at p if $\vec{H}(p) = 0$.

Example 5.1. The mean curvature vector of $f : (S^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$ is $\vec{H} = h_\eta \eta$, where η is the inward unit normal.

$$\forall p \in M \quad h_\eta(p) = \frac{1}{n} \operatorname{Tr}(S_\eta(p)) = \frac{1}{n} \operatorname{Tr}(id_{T_pS^n}) = 1$$

So $\vec{H} = \eta$

5.2 Hypersurface Example

Let $f : (M, g) \to (\widetilde{M}^{n+1}, \widetilde{g})$ be an isometric embedding, with $\eta \in \mathfrak{X}(M)^{\perp}$ being the unit normal vector field. This exists if and only if $N(f)$ is trivial.

Then with $S_\eta(p)$ being the self adjoint operator defined above, and taking an orthonormal basis $\{e_i\}_{i=1}^{n}$ of T_pM, we recall that $S_\eta(e_i) = \lambda_i e_i$

Then the eigenvalues $\lambda_1, \ldots, \lambda_n$ are principal curvatures of f at p. Also, the eigenvectors $e_1, \ldots, e_n$ are principal directions of f at p.

Define some symmetric functions on λ_i:

$$\begin{aligned}\sigma_1 &= \lambda_1 + \cdots + \lambda_n \\ \sigma_2 &= \sum_{i<j} \lambda_i \lambda_j \\ &\vdots \\ \sigma_n &= \lambda_1 \cdots \lambda_n\end{aligned}$$

Then $\sigma_2, \sigma_4, \ldots$ are invariants of the isometric embedding f.

And also:

$$h_\eta = \frac{1}{n}(\lambda_1 + \ldots + \lambda_n) \text{ is the mean curvature.}$$

$$\det(S_n) = \lambda_1 \cdots \lambda_n \text{ is the Gauss-Kronecker curvature}$$

Special Case:

Let M^2 be a 2 dimensional Riemannian manifold isometrically embedded into $(\mathbb{R}^3, dx^2 + dy^2 + dz^2)$.

For $p \in M$, $\eta \in (T_pM)^{\perp}$, there exists an orthonormal basis $e_1, \ldots, e_n$ on T_pM such that:

$$\begin{aligned}S_\eta(e_i) &= \lambda_i e_i \\ B(e_i, e_j) &= \lambda_i \delta_{ij} \eta\end{aligned}$$

Then we also have:

$$\begin{aligned}\mathbb{K}(p) &= \det(S_\eta) = \lambda_1 \lambda_2 \quad \text{ is the Gaussian curvature} \\ K(p) &= \widetilde{K}(p) + \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2 \\ &= 0 + \langle \lambda_1 \eta, \lambda_2 \eta \rangle - 0 = \lambda_1 \lambda_2 = \mathbb{K}(p)\end{aligned}$$

Theorem 5.1. *Gauss Theorema Egregium:*

The Guassian curvature of a 2 dimensional Riemannian manifold is an intrinsic invariant.

More generally, the Gauss-Kronecker curvature of an isometric embedding $M^{2n} \hookrightarrow \mathbb{R}^{2n+1}$ is an intrinsic invariant.

5.3 Gauss Map

Let $M^n \hookrightarrow (\mathbb{R}^{n+1}, g_0 = dx_1^2 + \ldots + dx_{n+1}^2)$

Suppose that there exists $N \in \mathfrak{X}(M)$ unit normal vector field $\forall p \in M^n$.

$$N(p) \in (T_pM)^\perp \subset T_p\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \qquad |N(p)| = 1$$

Then we obtain a C^∞ map $N : M^n \to S^n$, known as the *Gauss Map* of the isometric embedding $M^n \hookrightarrow (\mathbb{R}^{n+1}, g_0)$.

$$dN_p : T_pM \to T_{N(p)}S^n$$

Where $T_pM = \{v \in \mathbb{R}^{n+1} \mid \langle N(p), v\rangle = 0\} = T_{N(p)}S^n$

And then we have:

$$\forall v \in T_pM \quad S_{N(p)}(v) = -D_vN = -dN_p(v)$$

where D is the pullback connection of the Levi-Civita connection $\widetilde{\nabla}$ of $(\mathbb{R}^{n+1}, g_0)$.

Then from this, we see that if $x, y \in T_pM$, the second fundamental form:

$$H_{N(p)}(x, y) = \langle S_{N(p)}(x), y\rangle = -\langle dN_p(x), y\rangle$$

$$\begin{array}{ccc} U \subset M & \hookrightarrow & \mathbb{R}^{n+1} \\ {\scriptstyle\varphi}\downarrow & & \\ V \subset \mathbb{R}^n & & \end{array}$$

$$\mathbb{X}(\vec{u}) = \phi^{-1}(\vec{u}) = (X_1(\vec{u}), \ldots, X_{n+1}(\vec{u})) \in M \subset \mathbb{R}^{n+1}$$
$$\mathbb{N}(\vec{u}) = N(\mathbb{X}(\vec{u})) = (N_1(\vec{u}), \ldots, N_{n+1}(\vec{u})) \in S^n \subset \mathbb{R}^{n+1}$$

$$\mathbb{X} : V \to \mathbb{R}^{n+1} \quad C^\infty \text{ embedding}$$
$$\mathbb{N} : V \to \mathbb{R}^{n+1} \quad C^\infty \text{ map}$$

$$dN_p\left(\frac{\partial \mathbb{X}}{\partial u_i}\right) = \frac{\partial \mathbb{N}}{\partial u_i}$$

Let $(u_1, \ldots, u_n)$ be a local coordinate system on $U = \mathbb{X}(V) \subset M$. Then we have:

$$H_N = \sum_{i,j} h_{ij} du_i du_j \quad \text{where } h_{ij} = \langle \mathbb{X}_{ij}, N\rangle = -\langle \mathbb{X}_i, N_j\rangle$$
$$g = \sum_{i,j} g_{ij} du_i du_j \quad \text{where } g_{ij} = \langle \mathbb{X}_i, \mathbb{X}_j\rangle$$

5.4 Example:

Consider the surface of revolution obtained by rotating $y = \cosh(z)$ in the yz-plane about the z-axis. The parameterization is given by:

$$\begin{aligned}
&\mathbb{X} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3 \\
&\mathbb{X}(u, v) = (\cosh(v)\cos(u), \cosh(v)\sin(u), v)
\end{aligned}$$

Then first compute the partial derivatives:

$$\begin{aligned}
\mathbb{X}_u &= (-\cosh(v)\sin(u), \cosh(v)\cos(u), 0) \\
\mathbb{X}_v &= (\sinh(v)\cos(u), \sinh(v)\sin(u), 1)
\end{aligned}$$

Which allows us to solve for the components of the metric by leveraging the fact that $g_{ij} = \langle \mathbb{X}_i, \mathbb{X}_j \rangle$:

$$\begin{aligned}
g_{11} &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle = \cosh^2(v) \\
g_{12} &= \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0 \\
g_{22} &= \langle \mathbb{X}_v, \mathbb{X}_v \rangle = \cosh^2(v)
\end{aligned}$$

So that $g = (\cosh^2(v))(du^2 + dv^2)$

Now to solve for the unit normal vector field $\mathbb{N} = \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|}$, we have:

$$\begin{aligned}
\mathbb{X}_u \times \mathbb{X}_v &= \cosh(v)\langle \cos(u), \sin(u), -\sinh(v) \rangle \\
\mathbb{N} &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = \frac{\langle \cos(u), \sin(u), -\sinh(v) \rangle}{\cosh(v)}
\end{aligned}$$

And we can also compute the partial derivatives of $\mathbb{N}$:

$$\begin{aligned}
\mathbb{N}_u &= \frac{\langle -\sin(u), \cos(u), 0 \rangle}{\cosh(v)} = \frac{1}{\cosh^2(v)}\mathbb{X}_u \\
\mathbb{N}_v &= \frac{\langle 0, 0, -\cosh(v) \rangle \cosh(v) - \langle \cos(u), \sin(u), -\sinh(v) \rangle \sinh(v)}{\cosh^2(v)} = \frac{-1}{\cosh^2(v)}\mathbb{X}_v
\end{aligned}$$

Which shows that:

$$S_N = -dN = \frac{1}{\cosh^2(v)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ with respect to the basis } \frac{\partial}{\partial u}, \frac{\partial}{\partial v}$$

Principal Curvatures: $\lambda_1 = -\frac{1}{\cosh^2(v)}$, $\lambda_2 = \frac{1}{\cosh^2(v)}$

Principal Directions: $e_1 = \frac{1}{\cosh(v)}\frac{\partial}{\partial u}$, $e_2 = \frac{1}{\cosh(v)}\frac{\partial}{\partial v}$

Mean Curvature: $h_N = \frac{1}{2}(\lambda_1 + \lambda_2) = 0$

Gaussian Curvature: $K = \lambda_1\lambda_2 = -\frac{1}{\cosh^4(v)}$

Now to solve for $H_N = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2$, we have:

$$h_{11} = -\langle \mathbb{X}_u, \mathbb{N}_u \rangle = -\frac{1}{\cosh^2(v)} \langle \mathbb{X}_u, \mathbb{X}_u \rangle = -1$$
$$h_{12} = -\langle \mathbb{X}_u, \mathbb{N}_v \rangle = -\frac{1}{\cosh^2(v)} \langle \mathbb{X}_u, \mathbb{X}_v \rangle = 0$$
$$h_{22} = -\langle \mathbb{X}_v, \mathbb{N}_v \rangle = \frac{1}{\cosh^2(v)} \langle \mathbb{X}_v, \mathbb{X}_v \rangle = 1$$

Which gives us: $H_N = -du^2 + dv^2$

So $S \subset \mathbb{R}^3$ is a minimal surface, not totally geodesic.

Now let's compute the sectional curvature $K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}$:

Recall that the metric is as follows:

$$g_{11} = \cosh^2(v) \qquad g^{11} = \frac{1}{\cosh^2(v)}$$
$$g_{22} = \cosh^2(v) \qquad g^{22} = \frac{1}{\cosh^2(v)}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} \right) = 0$$
$$\Gamma^2_{11} = \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{21}}{\partial u} - \frac{\partial g_{11}}{\partial v} \right) = -\frac{1}{2} g^{22} \left(\frac{\partial}{\partial v} g_{11} \right)$$
$$= -\frac{1}{2\cosh^2(v)} \left(\frac{\partial}{\partial v} \cosh^2(v) \right) = -\frac{1}{2\cosh^2(v)} (2\cosh(v)\sinh(v)) = -\tanh(v)$$

Which implies that:

$$\nabla_{\frac{\partial}{\partial u}} \left(\frac{\partial}{\partial u} \right) = -\tanh(v) \frac{\partial}{\partial v}$$

And then:

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} g^{11} \left(\frac{\partial}{\partial v} g_{11} \right) = \tanh(v)$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \left(\frac{\partial}{\partial u} g_{22} \right) = 0$$

Which implies:

$$\nabla_{\frac{\partial}{\partial u}} \left(\frac{\partial}{\partial v} \right) = \nabla_{\frac{\partial}{\partial v}} \left(\frac{\partial}{\partial u} \right) = \tanh(v) \frac{\partial}{\partial u}$$

And finally:

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left(\frac{\partial}{\partial u} g_{22} \right) = 0$$
$$\Gamma^2_{22} = \frac{1}{2} g^{22} \left(\frac{\partial}{\partial v} g_{22} \right) = \tanh(v)$$

Which gives:

$$\nabla_{\frac{\partial}{\partial v}} \left(\frac{\partial}{\partial v} \right) = \tanh(v) \frac{\partial}{\partial v}$$

Now using these results, we can compute R_{1212} as:

$$\begin{aligned} R_{1212} &= \langle \nabla_v \nabla_u \partial_u - \nabla_u \nabla_v \partial_u, \partial_v \rangle \\ &= \langle \nabla_v(-\tanh(v)\partial_v) - \nabla_u(\tanh(v)\partial_u), \partial_v \rangle \\ &= \langle -\partial_v(\tanh(v))\partial_v - \tanh^2(v)\partial_v + \tanh^2(v)\partial_v, \partial_v \rangle \\ &= -\frac{1}{\cosh^2(v)} \langle \partial_v, \partial_v \rangle \\ &= -1 \end{aligned}$$

So that we finally achieve the final result:

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{-1}{\cosh^4(v)}$$

To summarize all of the information we have learned about this manifold:

Property	**Value**
Principal Curvatures	$\lambda_1 = -\frac{1}{\cosh^2(v)},\ \lambda_2 = \frac{1}{\cosh^2(v)}$
Principal Directions	$e_1 = \frac{1}{\cosh(v)} \frac{\partial}{\partial u},\ e_2 = \frac{1}{\cosh(v)} \frac{\partial}{\partial v}$
Mean Curvature	$h_N = \frac{1}{2}(\lambda_1 + \lambda_2) = 0$
Gaussian Curvature	$K = \lambda_1 \lambda_2 = -\frac{1}{\cosh^4(v)}$
Second Fundamental Form	$H_N = -du^2 + dv^2$
Sectional Curvature	$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{-1}{\cosh^4(v)}$

□

6 Complete Manifolds

From now on, let's assume that the underlying topological space of M is both Haussdorf and second countable.

6.1 Hadamard Theorem

Theorem 6.1. *(Hadamard Theorem)*

Let (M, g) be a complete Riemannian manifold, simply connected, and with sectional curvature $K(p, \sigma) \leq 0$, for all $p \in M$, and for all $\sigma \subset T_pM$. Then M is diffeomorphic to $\mathbb{R}^n$. More precisely, $\exp_p : T_pM \to M$ is a diffeomorphism for all $p \in M$.

6.2 Metric Spaces on Manifolds

Let (M, g) be a connected Riemannian manifold. Then for all $p, q \in M$, we define the *distance* between p and q to be:

$$d_g(p, q) = \inf\{\ell(c) \mid c : [0, 1] \to M \text{ is a piecewise smooth curve with } c(0) = p, c(1) = q\}$$

Which by definition implies that $0 \leq d_g(p, q) < \infty$. From now on, assume the metric g is fixed, so we can simply write $d(p, q)$.

Proposition 6.1. (M, d) is a metric space.

Proof:

By definition, we need to check the following properties:

(1) (Triangle Inequality) $d(p, r) \leq d(p, q) + d(q, r)$

(2) (Symmetry) $d(p, q) = d(q, p)$

(3) $d(p, q) \geq 0$

(4) $d(p, q) = 0 \iff p = q$

Notice that (1), (2), and (3) are obvious from the definition of $d(p, q)$. So we only need to prove (4).

If $p = q$, then it is also very clear to see that $d(p, q)$ must equal 0.

Now, suppose that $d(p, q) = 0$. We want to show that if $p \neq q$, then $d(p, q) > 0$, which will immediately imply the result.

Since M is Hausdorff, we know that there must exist an open neighborhood U of $p \in M$ such that $q \notin U$ for some $\epsilon > 0$. Moreover, we can choose U to be a normal neighborhood of p. Now, let $B = B_\epsilon(p)$ be a geodesic ball of radius $\epsilon > 0$ centered at $p \in M$, such that $\overline{B} \subset U$. Also let $\gamma : [0, 1] \to B$ be a geodesic line segment with $\gamma(0) = p$. Then if $c : [0, 1] \to M$ is a piecewise smooth curve with $c(0) = \gamma(0) = p$, $c(1) = \gamma(1)$, we know from do Carmo that this implies $\ell(\gamma) \leq \ell(c)$.

In the case when these lengths are equal, we must have that $c([0, 1]) = \gamma([0, 1])$.

From this it is easily concluded that $d(p, q) \geq r > 0$, which completes the proof.

$\square$.

Remark:

A non-Hausdorff space is not metrizable. For example, consider the line with two origins which is defined as the image of the map π. The equivalence relation $\sim$ is defined by $(x, 0) \sim (x, 1)$ for all $x \neq 0$.

$$\pi : \mathbb{R} \times \{0, 1\} \to M = \big(\mathbb{R} \times \{0, 1\}\big) / \sim$$

And notice that $d((0,0),(0,1)) = 0$, but $(0,0) \neq (0,1)$. Therefore, d is not a metric.

Also notice that if we fix $p_0 \in M$, then the function

$$\begin{aligned} f : M &\to \mathbb{R} \\ p &\mapsto d(p_0, p) \end{aligned}$$

is a continuous function.

$$|f(p) - f(q)| = |d(p_0, p) - d(p_0, q)| \leq d(p, q)$$

So that $f : (M, d) \to (\mathbb{R}, ||)$ is Lipschitz continuous.

Definition 6.2. (Geodesic Completeness)

A Riemannian manifold (M, g) is *geodesically complete* if for any $p \in M$, the exponential map $\exp_p(v)$is defined for all $v \in T_pM$, i.e., every geodesic $\gamma(t)$ is defined for all $t \in \mathbb{R}$.

Theorem 6.2. *(Hopf-Rinow Theorem)*

If (M, g) is a connected Riemannian manifold, and $p \in M$, define a metric d (in the sense of point set topology)on M as above. Then for the following statements:

(a) $\exp_p$ *is defined on* T_pM

(b) Closed and bounded sets of (M, d) are compact

(c) (M, d) is a complete metric space

(d) (M, g) is geodesically complete

(e) $\exists$ compact sents $K_n \subset M$, with $K_n \subset K_{n+1}$, with $\bigcup_{n=1}^{\infty} K_n = M$ such that $q_n \notin K_n \implies d(p, q_n) \to \infty$

(f) $\forall q \in M$ there exists a minimizing geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$

We have $(a) \iff (b) \iff (c) \iff (d) \iff (e) \implies (f)$.

Corollary 6.3. If M is a compact C^∞ manifold, then for any Riemannian metric g on M, (M, g) is geodesically complete.

But more generally, if we have an open embedding $M \overset{i}{\hookrightarrow} M'$ such that $i(M) \subset M'$ is a proper subset i.e., $i(M) \neq M'$, and (M', g') is a Riemannian manifold, then (M, i^*g) is not geodesically complete.

Definition 6.4. (Extendible Manifolds)

A connected Riemannian manifold (M, g) is said to be *extendible* if there exists a connected Riemannian manifold (M', g') such that $i : M \hookrightarrow M'$ is an open embedding, and:

$$i(M) \overset{\text{open}}{\subset} M' \quad \text{and} \quad i(M) \neq M' \quad i^*g' = g$$

Remark: Compact $\implies$ Complete $\implies$ Non-Extendible

Corollary 6.5. (Corollary of Hopf-Rinow)

Suppose that (M, g) is a connected complete Riemannian manifold. Let N be a closed submanifold of M, and $i : N \hookrightarrow M$ be an inclusion. Then:

(N, i^*g) is a complete Riemannian manifold.

6.3 Conjugate Points & Poles

Let (M, g) be a Riemannian manifold, and $\gamma : [0, a] \to M$ be a geodesic.

Choose $t_0 \in [0, a]$. We say that $\gamma(t_0)$ is *conjugate* to $\gamma(0)$ along γ if there is a Jacobi field J along γ such that $J(0) = J(t_0) = 0$, and J is not identically 0.

Let $\gamma'(0) = v \neq 0 \implies \gamma(t) = \exp_p(tv)$. Then:

$$J(0) = 0, \quad J'(0) = w \neq 0 \implies J(t) = (d\exp_p)_{tv}(tw)$$

Define the *multiplicity* of $\gamma(t_0)$ to be:

$$\begin{aligned} m(\gamma(t_0)) &= \dim\{J \mid J \text{ is a Jacobi field along } \gamma, J(0) = J(t_0) = 0\} \\ &= \dim\{w \in T_pM \mid (d\exp_p)_{t_0 v}(t_0 w) = 0\} \\ &= \dim\{\ker(d\exp_p)_{t_0 v}\} \end{aligned}$$

From the Gauss Lemma, we know that $|(d\exp_p)_{t_0 v}(v)| = |v| \neq 0$, which means that $v \notin \ker((d\exp_p)_{t_0 v})$. Because of this, we must have:

$$0 \leq \dim\ker((d\exp_p)_{t_0 v}) \leq n - 1$$

From this, we can also reformulate the statement as follows:

$\gamma(t_0)$ is conjugate to $\gamma(0)$ along $\gamma \iff t_0 v$ is a critical point of $\exp_p$

Definition 6.6. (Conjugacy Locus)

The *conjugacy locus* of $p \in M$ is the set of all (first) conjugate points along all geodesics $\gamma(t)$ in M with $\gamma(0) = p$. This set is denoted $C(p)$.

Definition 6.7. (Poles)

Let (M, g) be a connected complete Riemannian manifold. We say that $p \in M$is a *pole* if $C(p) = \emptyset$.

This is also equivalent to the following statements:

- $\forall v \in T_pM \quad (d\exp_p)_v : T_v(T_pM) \to T_{\exp_p(v)}M$ is a linear isomorphism.
- $\exp_p : T_pM \to M$ is a local diffeomorphism.

Intuitively, a pole is a point from which all geodesics emanate without conjugate points. In other words, a pole is a point from which all geodesics are minimizing. This means that the existence of a pole on a manifold (M, g) implies that it is possible to define a global coordinate system.

Lemma 6.3. *Suppose that (M, g) is a connected complete Riemannian manifold, with constant sectional curvature $K \leq 0$. Then this implies $\forall p \in M$, p is a pole.*

For this lemma, we define an arbitrary geodesic $\gamma : [0, \infty) \to M$, and impose the condition $\gamma(0) = p$.

Proof:

Let J be a Jacobi field along γ such that $J(0) = 0$ and $J'(0) \neq 0$. We want to prove that $J(t) \neq 0$ for $t \in (0, \infty)$. First, let's calculate $\langle J, J\rangle''$:

$$\begin{aligned}
\langle J, J\rangle'' &= (\langle J', J\rangle + \langle J, J'\rangle)' \\
&= (2\langle J', J\rangle)' \\
&= 2\langle J'', J\rangle + 2\langle J', J'\rangle \\
&= 2|J'(t)|^2 - 2\langle (R(J,\gamma')\gamma', J)\rangle
\end{aligned}$$

And recall from the definition of sectional curvature that

$$K(u,v) = \frac{\langle R(u,v)v, u\rangle}{|u|^2|v|^2 - \langle u,v\rangle^2} \implies \langle R(u,v)v,u\rangle = K(|u|^2|v|^2 - \langle u,v\rangle^2)$$

So that plugging this in, along with remembering that γ is orthogonal to J, implying $\langle J, \gamma'\rangle = 0$, we get:

$$\langle J, J\rangle'' = 2|J'(t)|^2 - 2K(J,\gamma')(|J|^2|\gamma'|^2)$$

But notice that we assumed that $K \leq 0$. This means that the expression can be rewritten as:

$$\langle J, J\rangle'' = 2|J'(t)|^2 + 2\alpha|J(t)|^2|\gamma'(t)|^2 \geq 0$$

where $\alpha = -K \geq 0$.

From this, we can conclude that $\langle J, J\rangle'$ is a non-decreasing function.

Now take $0 < t_1 < t_2$. Then by the fact that $\langle J, J\rangle'$ is non-decreasing, we have:

$$\langle J, J\rangle'(t_2) \geq \langle J, J\rangle'(t_1) \geq \langle J, J\rangle'(0) = 2\langle J'(0), J(0)\rangle = 2\langle J'(0), 0\rangle = 0$$

Which then implies $\langle J, J\rangle = |J(t)|^2$ is also non decreasing.

But:

$$\begin{aligned}
&J(0) = 0, J'(0) \neq 0 \\
&\qquad \implies \exists \delta > 0 \text{ such that } J(t) \neq 0 \text{ for } t \in (0,\delta) \\
&\qquad \implies |J(t)|^2 > 0 \text{ for } t \in (0,\delta) \\
&\qquad \implies |J(t)|^2 > 0 \text{ for } t \in (0,\infty)
\end{aligned}$$

But this means that $J(t)$ can never equal zero again, since $J(t)$ is a strictly positive function. Therefore, the conjugacy locus (Definition. 6.6). Then by definition 6.7, we see that the conjugate locus being empty means p is a pole.

Since p was an arbitrary point in the manifold M, we have shown that every point in M is a pole.

$\square$.

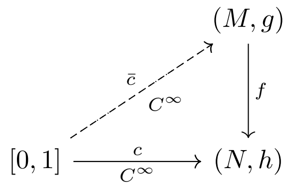
Lemma 6.4. *Let (M,g) be a connected complete Riemannian manifold. Also let (N,h) be a Riemannian manifold.*

If $f : M \to N$ is a surjective local diffeomorphism ($\implies$ N connected), then $\forall p \in M$, and $\forall v \in T_pM$, we have that $\left\|df_p(v)\right\|_{f(p)} \geq \|v\|_p \implies f$ is a covering map.

Proof:

It suffices to show that $f : \underset{\text{complete}}{(M, g)} \to (N, h)$ is a surjective local diffeomorphism.

Or that $\left\|(df_p)(v)\right\| \geq \|v\|$ has the path lifting property:



<u>Claim</u>

(1): If $\bar{c} : [0, t_0] \to M \quad 0 \leq t_0 < 1, \quad f \circ \bar{c} = c$

$\implies \exists \delta > 0$ such that $\bar{c}$ is defined on $[0, t_0 + \delta]$ and $f \circ \bar{c} = c$.

(2): If $\bar{c}$ is defined on $[0, t_0)$, $\quad 0 < t_0 \leq 1, \quad f \circ \bar{c} = c$, then $\bar{c}$ is defined at t_0.

<u>Proof of (1):</u>

We know that there must exist an open neighborhood V of $\bar{c}(t_0)$ in M such that $f|_V : V \subset M \to f(V) \subset N$ is a diffeomorphism.

This means that $f(V)$ is an open neighborhood of $c(t_0) = f \circ \bar{c}(t_0)$ in N.

Which implies that $\exists \delta > 0$ such that $|t - t_0| < \delta \implies c(t) \in f(V)$

Then define $\bar{c}(t) := (f|_V)^{-1}(c(t))$ for $t \in [t_0 - \delta, t_0 + \delta]$, so that $f \circ \bar{c}(t) = c(t) \quad t \in [0, t_0 + \delta)$

And we have arrived at our result. Note that we only used the fact that f is a diffeomorphism. $\square$

<u>Proof of (2):</u>

$\exists \{t_n\} \subset [0, t_0) \quad t_n < t_{n+1} \quad \lim_{n \to \infty} t_n = t_0$

Now choose $m < n$ so that:

$$\begin{aligned} d_M(\bar{c}(t_m), \bar{c}(t_n)) \leq \ell(\bar{c}|_{[t_m, t_n]}) &= \int_{t_m}^{t_n} \left\| \frac{d\bar{c}}{dt}(t) \right\|_{\bar{c}(t)} dt \\ &\leq \int_{t_m}^{t_n} \left\| df_{\bar{c}(t)} \left(\frac{d\bar{c}}{dt}(t) \right) \right\|_{\bar{c}(t)} dt \\ &= \int_{t_m}^{t_n} \left\| \frac{d(f \circ \bar{c})}{dt}(t) \right\|_{c(t)} dt \\ &\leq C(t_n - t_m) \end{aligned}$$

where $C = \max_{t \in [0,1]} \left\| \frac{dc}{dt}(t) \right\| > 0$

$\{\bar{c}(t_n)\}$ is a Cauchy sequence in (M, d_M) which is a complete metric space. By the assumption that (M, g) is geometrically complete, and Hopf-Rinow (Theorem 6.2),

$\implies r \in M$ such that $\lim_{n \to \infty} \bar{c}(t_n) = r$

Define $\bar{c}(t_0) = r$. Then:

$$f \circ \bar{c}(t_0) = f\left(\lim_{n\to\infty} \bar{c}(t_n)\right) = \lim_{n\to\infty} f \circ \bar{c}(t_n) = \lim_{n\to\infty} c(t_n) = c(t_0)$$

□

Corollary 6.8. If (M, g) is a connected complete Riemannian manifold, and $p \in M$ is a pole, then the exponential map $\exp_p : T_pM \to M$ is a covering map.

If, in addition, M is simply connected, then $\exp_p : T_pM \to M$ is a diffeomorphism, which implies M is diffeomorphic to $\mathbb{R}^n$.

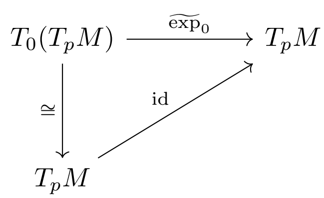
Proof:

By assumption, $\exp_p : T_pM \to M$ is a surjective local diffeomorphism. This implies that $g' = \exp_p^* g$ is a Riemannian metric on T_pM. This makes $\exp_p : (T_pM, g') \to (M, g)$ a local isometry.

By Lemma 6.4, it suffices to show that (T_pM, g') is complete.

$$\forall v \in T_0(T_pM) \cong T_pM \quad \gamma(t) = \exp_p(tv), t \in \mathbb{R} \text{ is a geodesic in } (M, g)$$

This implies that $\widetilde{\gamma}(t) = tv$, with $t \in \mathbb{R}$ is a geodesic in (T_pM, g')



Then by Hopf-Rinow (Theorem 6.2), we have that (T_pM, g') is complete.

□

Notice that by this Theorem, as well as Lemma 6.3, we have:

Theorem 6.5. *(Cartan-Hadamard Theorem)*

Suppose that (M, g) is a connected complete Riemannian manifold with $K(p, \sigma) \leq 0 \quad \forall p \in M$. $\sigma \in Gr(p, T_pM)$.

Then this implies that $\forall p \in M$, the map $\exp_p : T_pM \to M$ is a covering map.

In particular, if (M, g) is also simply connected, then $\forall p \in M$, the map $\exp_p : T_pM \to M$ is a diffeomorphism, which implies that M is diffeomorphic to $\mathbb{R}^n$.

7 Geodesics & Convex Neighborhoods

7.1 Geodesic Frame

Let (M, g) be a Riemannian manifold of dimension n, and $p \in M$, with $r > 0$ such that:

$$\exp_p : B_r(0) \subset T_pM \to B_r(p) \subset M$$

is a diffeomorphism.

Given any orthonormal basis of $(e_1(p), \dots, e_n(p))$ of T_pM, we define an orthonormal frame $(e_1, \dots, e_n)$ of $TM|_{B_r(p)} = TB_r(p)$ as follows:

$\forall q \in B_r(p) \quad \exists! v \in T_pM$ such that $\exp_p(v) = q$.

Then $\gamma : [0, 1] \to M$, with $\gamma(t) = \exp_p(tv)$ is a geodesic in (M, g), such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma'(0) = v$.

Let $V_i(t)$ be the unique parallel vector field along γ with the initial value $V_i(0) = e_i(p) \in T_pM$. Then we define:

$$e_i(q) := V_i(1) \in T_qM$$

Then:

- e_i is a C^∞ vector field on $B_r(p)$
- $\langle e_i(q), e_j(q)\rangle = \delta_{ij} \quad \forall q \in B_r(p)$
- $(\nabla_{e_i} e_j)(p) = 0$

We call $(e_1, \dots, e_n)$ the *geodesic frame* of $TM|_{B_r(p)}$. It is determined uniquely by $(e_1(p), \dots, e_n(p))$.

7.2 Theorem of Cartan

First, consider a local isometry $f : (M, g) \to (\widetilde{M}, \widetilde{g})$.

Then $\forall p \in M$, there exists $r > 0$ such that $f : B_r(p) \to B_r(\widetilde{p})$ $(\widetilde{p} = f(p))$ is an isometry.

$i := df_p : T_pM \to T_{\widetilde{p}}\widetilde{M}$ is a linear isometry, or isomorphism of inner product spaces.

$$\begin{array}{ccc} B_r(0) \subset T_pM & \xrightarrow{\quad i \quad} & B_r(0) \subset T_{\widetilde{p}}\widetilde{M} \\ \Big\downarrow {\scriptstyle \exp_p} & & {\scriptstyle \widetilde{exp}_{\widetilde{p}}} \Big\downarrow \\ B_r(p) & \xrightarrow{\quad f \quad} & B_r(\widetilde{p}) \end{array}$$

Where $f = \widetilde{\exp}_{\widetilde{p}} \circ i \circ (\exp_p)^{-1}$

Let $e_1(p), \dots, e_n(p)$ be an orthonormal basis of T_pM. Then let $\widetilde{e}_i(\widetilde{p}) := i(e_i(p))$. With this definition, we have that $\widetilde{e}_1(\widetilde{p}), \dots, \widetilde{e}_n(\widetilde{p})$ is an orthonormal basis of $T_{\widetilde{p}}\widetilde{M}$.

Now let $e_1, \dots, e_n$ and $\widetilde{e}_i, \dots, \widetilde{e}_n$ be the geodesic frames on $U := B_r(p)$ and $\widetilde{U} := B_r(\widetilde{p})$, respectively. These are determined uniquely by $(e_1(p), \dots, e_n(p)) \in T_pM$ and $(\widetilde{e}_1(\widetilde{p}), \dots, \widetilde{e}_n(\widetilde{p})) \in T_{\widetilde{p}}\widetilde{M}$.

$$\begin{array}{ccc} U \times \mathbb{R}^n & \xrightarrow{f \times \mathrm{id}} & \widetilde{U} \times \mathbb{R}^n \\ \uparrow h & \circlearrowleft & \uparrow \widetilde{h} \\ TU & \xrightarrow{df} & T\widetilde{U} \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & \widetilde{U} \end{array}$$

$$h\left(q, \sum_{j=1}^{n} c_j e_j(q)\right) = (q, (c_1, \ldots, c_n))$$
$$\widetilde{h}\left(\widetilde{q}, \sum_{j=1}^{n} c_j \widetilde{e}_j(\widetilde{q})\right) = (\widetilde{q}, (c_1, \ldots, c_n))$$
$$df : TU \to T\widetilde{U} \quad (q, w) \mapsto (f(q), df_q(w))$$

Where $df_q(e_j(q)) = \widetilde{e}_j(\widetilde{q})$, $q \in U$, $w \in T_qM = T_qU$.

Now define the Riemann curvature tensor on both U and $\widetilde{U}$ to be:

$$R_{ijk\ell} := R(e_i, e_j, e_k, e_\ell) \in C^\infty(U)$$
$$\widetilde{R}_{ijk\ell} := \widetilde{R}(\widetilde{e}_i, \widetilde{e}_j, \widetilde{e}_k, \widetilde{e}_\ell) \in C^\infty(\widetilde{U})$$

Then we have that $R_{ijk\ell} = f^* \widetilde{R}_{ijk\ell}$, which implies that $\forall q \in U$, $R_{ijk\ell}(q) = \widetilde{R}_{ijk\ell}(f(q))$.

This is true if and only if $\forall q \in U$, and $\forall x, y, u, v \in T_qM$,

$$R(q)(x, y, u, v) = \widetilde{R}(f(q))(df_q(x), df_q(y), df_q(u), df_q(v))$$

Theorem 7.1. *(E. Cartan)*

Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds of the same dimension. Also let $p \in M$ and $\widetilde{p} \in \widetilde{M}$.

$$i : T_pM \to T_{\widetilde{p}}\widetilde{M} \text{ is a linear isometry.}$$

Then $\exists r > 0$ such that $f := \widetilde{\exp}_{\widetilde{p}} \circ i \circ \exp_p^{-1} : U = B_r(p) \to \widetilde{U} = B_r(\widetilde{p})$ is a diffeomorphism.

Let $(e_1, \ldots, e_n)$ be a geodesic frame on $B_r(p)$, and $(\widetilde{e}_1, \ldots, \widetilde{e}_n)$ be a geodesic frame on $B_r(\widetilde{p})$.

Then $\widetilde{e}_i(p) = i(e_i(p))$

$$\text{If } R_{ijk\ell}(q) = \widetilde{R}_{ijk\ell}(f(q)) \quad \forall q \in B_r(p), \text{ then } f \text{ is an isometry.} \tag{7.1}$$

Remark 7.2. For all $q \in U$, define a linear isomorphism

$$\begin{aligned} \phi_q : T_qM &\to T_{f(q)}\widetilde{M} \\ \sum_{i=1}^{n} c_i e_i(q) &\mapsto \sum_{i=1}^{n} c_i \widetilde{e}_i(f(q)) \end{aligned}$$

Then ϕ_q is a linear isometry, and ϕ_q is determined by i.

Equation 7.1 $\iff$ $\forall q \in U, \forall x, y, u, v \in T_qM$, we have that:

$$R(q)(x,y,u,v) = \widetilde{R}(f(q))(\phi_q(x), \phi_q(y), \phi_q(u), \phi_q(v))$$

And once we prove that f is an isometry, we immediately know that $\phi_q = df_q$.

Corollary 7.1. If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds of the same dimension, and same constant sectional curvature K_0

Let $p \in M$ and $\widetilde{p} \in \widetilde{M}$ be *any* point, let $i : T_pM \to T_{\widetilde{p}}\widetilde{M}$ be *any* linear isometry.

Then there exists:

- an open neighborhood of V of $p \in M$
- an open neighborhood $\widetilde{V}$ of $\widetilde{p} \in \widetilde{M}$
- an isometry $f : V \to \widetilde{V}$ such that $f(p) = \widetilde{p}$ and $df_p = i$

7.3 Conformal Deformation of Curvature

Suppose that g is a Riemannian metric on a manifold M, and let $\widetilde{g} = e^{2f}g$ for any smooth function f on M. Let ∇ and $\widetilde{\nabla}$ be connections on (M, g) and $(M, \widetilde{g})$, respectively.

Then we have that for any C^∞ vector fields $X, Y \in \mathfrak{X}(M)$

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X,Y)\,\mathrm{grad}_g f$$

We can prove this by assuming that $\widetilde{\nabla}$ has the form $\widetilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$, where A is a symmetric bilinear tensor. Plugging this in, and applying some elementary definitions, we can arrive at the result.

In particular, if f is constant, so that $\widetilde{g} = r^2 g$, then $\widetilde{\nabla}_X Y = \nabla_X Y$, which implies the following:

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z \\ \widetilde{R}(X,Y,Z,W) &= r^2 R(X,Y,Z,W) \\ \widetilde{Ric} &= Ric \\ \widetilde{S} &= r^{-2} S \end{aligned}$$

Definition 7.2. Given two symmetric $(0, 2)$-tensors S, and T on M, we can define the *Kulkarni-Nomizu product* of S and T to be the $(0, 4)$-tensor $S \otimes T$ defined by:

$$\begin{aligned} (S \owedge T)(X,Y,Z,W) = &S(X,Z)T(Y,W) + S(Y,W)T(X,Z) \\ &- S(X,W)T(Y,Z) - S(Y,Z)T(X,W) \end{aligned}$$

In particular:

$$(S \owedge S)(X,Y,Z,W) = 2(S(X,Z)S(Y,W) - S(X,W)S(Y,Z))$$

(M,g) has constant sectional curvature $K_0 \iff R = \frac{1}{2}K_0 g \owedge g$

Lemma 7.3. *For any $X,Y,Z,W \in \mathfrak{X}(M)$, we have:*

$$\begin{aligned}(S \owedge T)(X,Y,Z,W) &= -(S \owedge T)(Y,X,Z,W) \\ &= -(S \owedge T)(X,Y,W,Z) \\ &= (S \owedge T)(Z,W,X,Y)\end{aligned}$$

i.e., $S \owedge T \in C^\infty\left(M, \mathrm{Sym}^2\left(\bigwedge^2 T^*M\right)\right)$

Recall that $R \in C^\infty\left(M, \mathrm{Sym}^2\left(\bigwedge^2 T^*M\right)\right)$. Therefore, we can prove that:

$$\widetilde{R} = e^{2f}\left(R - (\mathrm{Hess}\, f) \owedge g + (df \otimes df) \owedge g - \frac{1}{2}|df|^2 g \owedge g\right)$$

Where:

$$\begin{aligned}\mathrm{Hess}(f) &= \sum_{ij} f_{,ij}\, dx_i dx_j \\ |df|^2 &= \sum_{ij} g^{ij} f_{,i} f_{,j}\end{aligned}$$

7.4 Example: Hyperbolic Space

Recall that:

$$\widetilde{R} = e^{2f}\left(R - (\mathrm{Hess}\, f) \owedge g + (df \otimes df) \owedge g - \frac{1}{2}|df|^2 g \owedge g\right)$$

Where $\owedge$ is defined as in Definition 7.2.

7.4.1 Upper Half Space Model

Let $H^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$

And then define a metric $\widetilde{g}$ on H^n by:

$$\widetilde{g} = \frac{dy_1^2 + \ldots + dy_n^2}{y_n^2} = e^{2f} g \quad g = dy_1^2 + \ldots + dy_n^2$$

Then:

$$
\begin{aligned}
R = 0, \quad e^f = \frac{1}{y_n} &\implies f = -\log y_n \\
f_i = \frac{\partial f}{\partial y_i} = -\delta_{in} &\implies df = -\frac{dy_n}{y_n} \\
f_{ij} = \frac{\partial^2 f}{\partial y_i \partial y_j} = \delta_{in}\delta jn &\implies \text{Hess} f = \sum_{ij} f_{ij} dy_i dy_j = \frac{dy_n^2}{y_n^2} \\
|df|^2 = \frac{1}{y_n^2} &= e^{2f} \\
df \otimes df = \frac{dy_n^2}{y_n^2} &= \text{Hess} f
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\widetilde{R} &= e^{2f}\left(R - (\text{Hess} f) \owedge g + (df \otimes df) \owedge g - \frac{1}{2}|df|^2 g \owedge g\right) \\
&= e^{2f}\left(0 - (\text{Hess} f) \owedge g + (\text{Hess} f) \owedge g - \frac{1}{2}|df|^2 g \owedge g\right) \\
&= -\frac{1}{2}e^{4f} g \owedge g \\
&= -\frac{1}{2}\widetilde{g} \owedge \widetilde{g} \\
&\implies (H^n, \widetilde{g}) \text{ has constant sectional curvature -1}
\end{aligned}
$$

7.4.2 Disk Model

Let $D^n = \{\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n \mid |\vec{u}| < 1\}$, which is the open unit ball in $\mathbb{R}^n$.

Define a metric:

$$
\widetilde{g} = \frac{4}{(1 - |\vec{u}|^2)^2}\left(du_1^2 + \ldots + du_n^2\right) = e^{2f} g \quad g = du_1^2 + \ldots + du_n^2
$$

So that:

$$
\begin{aligned}
|\vec{u}| &= \sqrt{u_1^2 + \ldots + u_n^2} \\
e^f &= \frac{2}{1 - |\vec{u}|^2} \\
f &= \log\left(\frac{2}{1 - |\vec{u}|^2}\right) \\
f_i &= \frac{2u_i}{1 - |\vec{u}|^2} \\
f_{ij} &= \frac{2\delta_{ij}}{1 - |\vec{u}|^2} + \frac{4u_i u_j}{(1 - |\vec{u}|^2)^2}
\end{aligned}
$$

Using these equivalences, we can see that:

$$
\begin{aligned}
df &= \frac{2\sum_i u_i du_i}{1-|\vec{u}|^2} \\
df \otimes df &= \frac{4\sum_{i,j} u_i u_j du_i du_j}{(1-|\vec{u}|^2)^2} \\
\text{Hess}(f) &= \sum_{i,j} f_{ij} du_i du_j = \frac{2\sum_i du_i^2}{1-|\vec{u}|^2} + \frac{4\sum_{i,j} u_i du_i u_j du_j}{(1-|\vec{u}|^2)^2} \\
|df|^2 &= \frac{4|\vec{u}|^2}{(1-|\vec{u}|^2)^2}
\end{aligned}
$$

And plugging all of this into the expression for $\widetilde{R}$, we can see that:

$$
\begin{aligned}
\widetilde{R} &= e^{2f}\left(R - (\text{Hess } f) \owedge g + (df \otimes df) \owedge g - \frac{1}{2}|df|^2 g \owedge g\right) \\
&= e^{2f}\left(0 - (\text{Hess } f) \owedge g + (df \otimes df) \owedge g - \frac{1}{2}|df|^2 g \owedge g\right) \\
&= e^{2f}\left((df \otimes df - \text{Hess } f) \owedge g - \frac{2|\vec{u}|^2}{(1-|\vec{u}|^2)^2} g \owedge g\right) \\
&= e^{2f}\frac{-2}{(1-|\vec{u}|^2)^2} g \owedge g \\
&= -\frac{1}{2}e^{4f} g \owedge g \\
&= -\frac{1}{2}\widetilde{g} \owedge \widetilde{g} \\
&\implies (D^n, \widetilde{g}) \text{ has constant sectional curvature -1}
\end{aligned}
$$

From this, we can conclude that (D^n, h) and (H^n, g) are isometric, and have constant sectional curvature -1.

Proposition 7.3. (D^n, h) is complete ($\iff$ (H^n, g) is complete)

Proof:

By the Hopf-Rinow Theorem (6.2), it suffices to show that $\exp_0$ is defined on T_0D^2.

For all $A \in O(n)$, let's define a function:

$$
\begin{aligned}
\phi_A : \mathbb{R}^n &\to \mathbb{R}^n \\
\vec{u} &\mapsto \vec{u}A
\end{aligned}
$$

So that the following holds:

$$\phi_A^* u_i = \sum_{j=1}^n u_j A_{ji} \qquad \phi_A^* du_i = \sum_{j=1}^n du_j A_{ji}$$

$$\phi_A^* \left(\sum_{i=1}^n u_i^2 \right) = \sum_{i=1}^n u_i^2 \qquad \phi_A^* \left(\sum_{i=1}^n du_i^2 \right) = \sum_{i=1}^n du_i^2$$

We also see that $\phi_A(D^n) = D^n$ and $\phi_A^* h = h$.

The differential:

$$\begin{aligned} (d\phi_A)_{\vec{u}} : T_{\vec{u}} D^n \cong \mathbb{R}^n &\to T_{\vec{u}A} D^n \cong \mathbb{R}^n \\ v &\mapsto vA \end{aligned}$$

Also for all unit tangent vectors $\vec{v} \in T_0 D^n$ there exists $A \in O(n)$, where $O(n)$ is the orthogonal group of $n \times n$ matrices such that $\vec{v}A = \frac{1}{2}\frac{\partial}{\partial u_1} \in T_0 D^n \cong \mathbb{R}^n = (\frac{1}{2}, 0, \dots, 0)$

It only remains to show that the normalized geodesic $\gamma(t)$ in (D^n, h) with $\gamma(0) = 0$, $\gamma'(0) = \frac{1}{2}\frac{\partial}{\partial u_1}$ is defined $\forall t \in \mathbb{R}$.

Let $\sigma : D^n \to D^n$ such that $\sigma(u_1, \dots, u_n) = (u_1, -u_2, \dots, -u_n)$

Then σ is an isometric involution on (D^n, h).

$(D^n)^\sigma = \{(u_1, 0, \dots, 0) \mid u_1 \in (-1, 1)\} = (-1, 1) \times \{(0, \dots, 0)\}$

It is then possible to prove that D^σ is a totally geodesic submanifold of (D^n, h), and that the induced metric on $D^\sigma \cong (-1, 1)$ is $\frac{4du_1^2}{(1-u_1^2)^2}$.

Now define $\beta : (-1, 1) \to (D^n)^\sigma$ by $\beta(t) = (t, 0, \dots, 0)$. Also let t_0 be an arbitrary point in $(-1, 1)$.

$$\begin{aligned} s(t_0) &:= \ell(\beta|_{[0,t_0]}) \\ &= \int_0^{t_0} |\beta'(t)|_h dt \\ &= \int_0^{t_0} \frac{2}{1-t^2} dt \\ &= \int_0^{t_0} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) dt \\ &= \log \left(\frac{1+t_0}{1-t_0} \right) \end{aligned}$$

So from this, $s = \log(\frac{1+t}{1-t})$, meaning $e^s = \frac{1+t}{1-t}$

From this equality, with some simple rearranging, we see:

$$\begin{aligned} \tanh\left(\frac{s}{2}\right) &= \frac{e^{\frac{s}{2}} - e^{-\frac{s}{2}}}{e^{\frac{s}{2}} + e^{-\frac{s}{2}}} = \frac{e^s - 1}{e^s + 1} \\ &= \frac{2t}{t} \\ &= t \end{aligned}$$

So $\gamma(s) = (\tanh(\frac{s}{2}), 0, \ldots, 0)$ is a normalized geodesic in (D^n, h) with $\gamma(0) = 0$, $\gamma'(0) = \frac{1}{2}\frac{\partial}{\partial u_1}$.

Therefore, we can determine the formula for the exponential map as:

$$\exp_{\vec{0}}(\vec{a}) = \begin{cases} \vec{0} & \text{if } \vec{a} = \vec{0} \\ \tanh\left(|\vec{a}|\right) \frac{\vec{a}}{|\vec{a}|} & \text{if } \vec{a} \neq \vec{0} \end{cases}$$

Which is obviously defined for all $\vec{a} \in T_0 D^n$.

$\square$

7.5 Möbius transform

$\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})/\{\pm(I_2)\}$ acts on $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

$\mathrm{Aut}(\mathbb{CP}^1) = \{\phi : \mathbb{CP}^1 \to \mathbb{CP}^1 \mid \phi \text{ is biholomorphic}\} = \mathrm{PSL}(2, \mathbb{C}) \supset \mathrm{PSL}(2, \mathbb{R})$

Then by do Carmo [dC] p.46, exercise 4, we have that $\mathrm{PSL}(2, \mathbb{R})$ acts isometrically on $(H^2, \frac{dx^2 + dy^2}{y^2})$

8 Space Forms

Lemma 8.1. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds, and M connected.*

Suppose the functions:

$$f_1, f_2 : (M, g) \to (\widetilde{M}, \widetilde{g})$$

are local isometries.

Then if there exists a $p \in M$ such that:

$$\begin{aligned} f_1(p) &= f_2(p) = \widetilde{p} \in \widetilde{M} \\ df_1(p) &= df_2(p) = i : T_pM \to T_{\widetilde{p}}\widetilde{M} \end{aligned}$$

Then $f_1 = f_2$.

Corollary 8.1. Let (M, g) be a connected Riemannian manifold, and G be a subgroup of $\text{Isom}(M, g) = \{\phi : M \to M \mid \phi \in C^\infty(M), \phi^* g = g\}$ such that:

- G acts transitively on M
- $p \in M$, and $G_p = \{\phi \in G \mid \phi(p) = p\} \to O(n)$ is a subgroup of G such that it maps the value $\phi \mapsto d\phi_p : T_pM \to T_pM$.

Then $G = \text{Isom}(M, g)$

We assume that $G_p \mapsto O(n)$ is a group isomorphism.

Example 8.1. $(\mathbb{R}^n, g_0 = dx_1^2 + \ldots + dx_n^2)$

Note that $O(n) \ltimes \mathbb{R}^n$ acts transitively and isometrically on $(\mathbb{R}^n, g_0)$.

$(A, \vec{b})\vec{x} = A\vec{x} + \vec{b}$

Notice that the stabilizer of $\vec{0}$ is $O(n)$.

Therefore, by the corollary above, we have that $\text{Isom}(\mathbb{R}^n, g_0) = O(n) \ltimes \mathbb{R}^n$, which represents rigid motion.

Also, $\text{Isom}(\mathbb{R}^n, g_0) = SO(n) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n = SO(n) \ltimes \mathbb{R}^n / SO(n)$

Using similar methods, we can also derive the following equivalences:

$$\begin{aligned} \text{Isom}(S^n, g_{\text{can}}) &= O(n+1) \\ \text{Isom}_0(S^n, g_{\text{can}}) &= SO(n+1) \\ \text{Isom}(H^2, g) &= \text{PSL}(2, \mathbb{R}) \sqcup \sigma\, \text{PSL}(2, \mathbb{R}) \quad (\sigma(x, y) = (-x, y)) \\ \text{Isom}_0(H^2, g) &= \text{PSL}(2, \mathbb{R}) \\ \text{Isom}(D^2, h) &= \text{PSU}(1, 1) \sqcup \sigma\, \text{PSU}(1, 1) \\ \text{Isom}_0(D^2, h) &= \text{PSU}(1, 1) \end{aligned}$$

8.1 Space Forms

A *space form* is a connected complete Riemannian manifold with constant sectional curvature.

Theorem 8.2. *Let (M,g) be a connected complete Riemannian manifold of dimension $n \geq 2$, with constant sectional curvature K.*

Let $(\widetilde{M},\widetilde{g})$ be the universal cover of (M,g)

Then:

$$(\widetilde{M},\widetilde{g}) \text{ is isometric to } \begin{cases} (H^n, g) & \text{if } \kappa = -1, \\ (\mathbb{R}^n, g_0) & \text{if } \kappa = 0, \\ (S^n, g_{can}) & \text{if } \kappa = 1. \end{cases}$$

This also implies that $K_{\lambda^2 g} = \frac{1}{\lambda^2} K_g$

Proposition 8.2. If M is a space form with $K > 0$, and $n := \dim M$ is even, then M is isometric to S^n or $P_n(\mathbb{R})$.

In particular, if M is orientable, then $M \cong S^n$.

Proof:

$M = S^{2m}/\Gamma$ where Γ is a finite subgroup of $\text{Isom}(S^{2m}, g_{\text{can}}) = O(2m+1)$. We see that Γ acts freely on S^{2m}.

Let $\phi \in \Gamma$.

The eigenvalues of Γ are then:

$$\{e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_k}, e^{-i\theta_k}, 1, \dots, 1, -1, \dots, -1\}$$

Where $\theta_i \in (0,\pi)$, and there are r 1's, s -1's, so that $2k + r + s = 2m + 1$ ($k, r, s \in \mathbb{Z}_{\geq 0}$)

And $\det(\phi) = (-1)^s$

<u>Case 1: $r > 0$</u>

$\exists$ a unit vector $\vec{x} \in S^{2m} \subset \mathbb{R}^{2m+1}$ such that $\phi(\vec{x}) = \vec{x} \implies \phi = id_{S^{2m}}$.

Therefore, Γ acts freely on S^{2m}

<u>Case 2: $r = 0$</u>

The eigenvalues of $\phi^2 \in \Gamma$ are:

$$\{e^{2i\theta_1}, e^{-2i\theta_1}, \dots, e^{2i\theta_k}, e^{-2i\theta_k}, 1, \dots, 1\}$$

where there are $2m + 1 - 2k$ 1's.

$\phi^2 = id_{S^{2m}} \implies \phi = -id_{S^{2m}}$ because there are $2m + 1$ eigenvalues of ϕ, and all the eigenvalues are -1. Raising $(-1)^{2m+1}$ shows that we must have $\phi = -id_{S^{2m}}$.

$$\Gamma = \{I_{2m+1}\} \implies M = S^{2m} = S^n$$

or

$$\Gamma = \{\pm I_{2m+1}\} \implies M = S^{2m}/\{\pm I_{2m+1}\} = P_n(\mathbb{R})$$

□

8.2 Conformal Maps

Definition 8.3. Let V, W be finite dimensional inner product spaces. A linera map $L : V \to W$ is called *conformal* if it is a linear isomorphism and:

$$\frac{\langle L(v_1), L(v_2)\rangle_W}{|L(v_1)|_W \cdot |L(v_2)|_W} = \frac{\langle v_1, v_2\rangle}{|v_1|_W \cdot |v_2|_W} \quad \forall v_1, v_2 \in V - \{0\}$$

So that L preserves unoriented angles.

Lemma 8.3. *Let $L : V \to W$ be a linear isomorphism between finite dimensional inner product spaces. Then the following are equivalent:*

1. *L is conformal*
2. *$\exists \lambda > 0$ such that $|L(v)|_W = \lambda |v|_V \quad \forall v \in V$*
3. *$\exists \lambda > 0$ such that $\langle L(v_1), L(v_2)\rangle_W = \lambda^2 \langle v_1, v_2\rangle_V \quad \forall v_1, v_2 \in V$*

Definition 8.4. Let (M, g) and (N, h) be Riemannian manifolds. A C^∞ map $f : M \to N$ is *conformal* if $\forall p \in M$, we have that $df_p : T_pM \to T_{f(p)}N$ is conformal.

$\iff$

f is a local diffeomorphism and $f^*h = \lambda^2 g$ for some C^∞ function $\lambda : M \to (0, \infty)$. The function λ^2 is called the conformal factor.

Note that a local isometry is simply a conformal map with $\lambda = 1$. We also have the following:

local isometry $\implies$ conformal $\implies$ local diffeomorphism.

Example 8.2. (Dilation)

$f : \mathbb{R}^n \to \mathbb{R}^n$, where $f(\vec{x}) = \lambda\vec{x}$, where $\lambda > 0$.

Under this map, we see that $g_0 = dx_1^2 + \ldots + dx_n^2$, and $f^*dx_i = \lambda dx_i$ so that $f^*g_0 = \lambda^2 g_0$. We can also see that $\forall \vec{x} \in \mathbb{R}^n$, $\det(df_{\vec{x}}) = \lambda^n > 0$.

So f is an orientation preserving conformal map from $(\mathbb{R}^n, g_0)$ to $(\mathbb{R}^n, g_0)$

Theorem 8.4. *(Liouville)*

Let $f : U \to \mathbb{R}^n$ be a conformal map, where U is connected, and $n \geq 3$.

Then this implies f is the restriction to U of a composition of isometries, dilation, or inversion, at most one of each.

Let G be the group generated by isometries, dilations, and inversions. Then

$$G = PSL(2, \mathbb{C}) \sqcup \sigma PSL(2, \mathbb{C})$$

Where $\sigma(z) = \bar{z}$ and $\sigma(x, y) = (x, -y)$.

Theorem 8.5. *(do Carmo [dC] p. 175, Thm 5.3)*

The isometries of H^n where $n \geq 2$ are restrictions to $H^n \subset \mathbb{R}^n$ of the conformal transformations of $\mathbb{R}^n$ that take H^n onto itself.

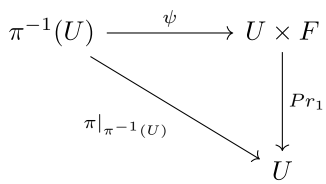
9 Principal Bundles

9.1 Definitions & Examples

Definition 9.1. Let E, M, and F be smooth manifolds, with $\pi : E \to M$ a smooth map. We say that (π, E, M) is a C^∞ **fiber bundle** with **total space** E and **fiber** F if:

(1): π is surjective

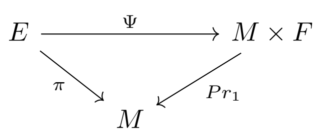
(2): $\forall x \in M$, there exists an open neighborhood $x \in U \subset M$ and a C^∞ diffeomorphism ψ such that the following diagram commutes:



where Pr_1 is the projection onto the first factor. These maps ψ are called **local trivializations** of our fiber bundle.

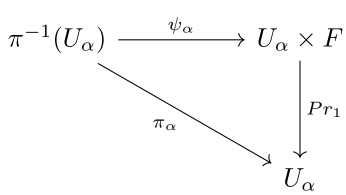
For example, in the case when $E = M \times F$, with $\pi : E = M \times F \to M$ given by Pr_1 is the **product** fiber bundle with base M and fiber F.

Also, a fiber bundle is **trivial** if there exists a diffeomorphism $\Psi : E \to M \times F$ such that the following diagram commutes:



Remark:

For a fiber bundle (π, E, M), there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of the base M and maps ψ_a that makes the diagram:



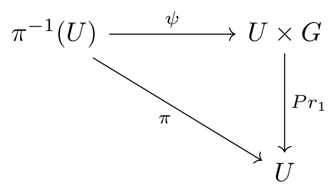
commute such that for $U_\alpha \cap U_\beta \neq \emptyset$, the map $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is given by $(\psi_\alpha \circ \psi_\beta^{-1})(x, \xi) = (x, \phi_x(\xi))$ where $\phi_x : F \to F$ is a smooth diffeomorphism.

Example 9.1. A C^∞ real (complex) vector bundle of rank r over M is a fiber bundle with fiber $\mathbb{R}^r(\mathbb{C}^r)$ such that $\phi_x : F \to F$ is an $\mathbb{R}$-linear ($\mathbb{C}$-linear) isomorphism. Note that this is much more restrictive! If we write $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$ as $(x, v) \mapsto (x, \psi_{\alpha\beta}(v))$, we have that the smooth map $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$, which is the same as saying $\phi_x v = Av$ for some $A \in GL(r, \mathbb{R}^r)$.

Definition 9.2. (Principal Bundle)

Let P and M be smooth manifolds, let G be a Lie group, and consider a smooth map $\pi : P \to M$. We say that (π, P, M, G) is a **principal fiber bundle** with total space P, base space M, and **structure group** G if:

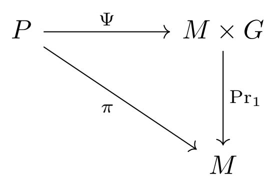
1. G acts smoothly and freely on P on the right
2. $\pi : P \to M$ is the natural projection $P \to P/G$
3. $\forall x \in M$ there exists an open neighborhood $x \in U \subset M$ and smooth diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that



commutes **and** $\psi(p \cdot g) = \psi(p) \cdot g$ for all $p \in \pi^{-1}(U)$ and all $g \in G$. Here, G acts on $U \times G$ on the right by $(x, u) \cdot g = (x, u \cdot g)$. These two conditions can be summarized by saying that we have G-equivariant local trivializations. We can also refer to this bundle $\pi : P \to M$ as simply a principal G-bundle.

Example 9.2. The **product** principal bundle is the case when $P = M \times G$, with $\pi : P = M \times G \to M$ given by Pr_1 and $(x, h) \cdot g = (x, hg)$, for all $x \in M$ and $g, h \in G$.

Example 9.3. A principal G-bundle is called **trivial** if there exists a C^∞ diffeomorphism $\Psi : P \to M \times G$ such that $\Psi(p \cdot g) = \Psi(p) \cdot g \quad \forall p \in P, g \in G$ such that the following diagram commutes:



Example 9.4. Given $\pi : E \to M$ a real vector bundle of rank r, let us describe the **frame bundle** $GL(E)$ of E. This is:

$$GL(E) = \{(x, (e_1, \dots, e_r)) \mid x \in M, (e_1, \dots, e_r) \text{ is an ordered } \mathbb{R}\text{-basis of } E_r\}$$

Let $\pi : GL(E) \to M$ be the projection Pr_1, and note that $GL(r, \mathbb{R})$ acts on $GL(E)$ on the right by:

$$(x, (e_1, \dots, e_r)) \cdot A = \left(x, \sum_{i=1}^{n} e_i A_{i1}, \dots, \sum_{i=1}^{n} e_i A_{ir} \right)$$

Where $A = (A_{ij}) \in GL(r, \mathbb{R})$, and $(x, (e_1, \dots, e_r)) \in GL(E)$. Then in this case, we can see that once we convince ourselves that $GL(E)$ is a manifold, $GL(E)$ is a principal $GL(r, \mathbb{R})$-bundle over M.

Additionally, if h is a metric on E, i.e., for all $x \in M$, $h(x)$ is an inner product on E_x, then h is a C^∞ section of $E^* \otimes E^*$.

We can also consider:

$$O(E, h) := \{(x, (e_1, \dots, e_r)) \in GL(E) \mid (e_1, \dots, e_r) \text{ is an ordered orthonormal basis of } (E_x, h(x))\}$$

Which is $U(E, h)$ if we are working over $\mathbb{C}$ instead of $\mathbb{R}$.

So, $O(E, h) \to M$ is a principal $O(n)$-bundle, and $U(E, h) \to M$ is a principal $U(n)$-bundle.

Associate Bundles:

Given a principal G-bundle $\pi : P \to M$, and a smooth manifold F that admits a left-action, G acts freely on $P \times F$ by:

$$(p, \xi) \cdot g = (p \cdot g, g^{-1} \cdot \xi) \quad \text{for } p \in P, \xi \in F, g \in G$$

Consider the space $P \times_G F := (P \times F)/G$ with the projection $P \times_G F \to M$ given by $[p, \xi] \mapsto \pi(p)$. Extending the local trivializations $U \times G$ of P to $U \times G \times F$, we see that the G action glues together $(x, h, \xi) \sim (x, 1, h^{-1}\xi)$, in which case, $P \times_G F \to M$ is a fiber bundle with fiber F known as the associate bundle of $\pi : P \to M$ with respect to F.

We can further generalize thie example. Given a principal G-bundle $\pi : P \to M$, and $\rho : G \to GL(n, \mathbb{R})$, a real representation of G, G acts on $\mathbb{R}^n$ on the left by $g \cdot v = \rho(g)v$ for $v = [v_1, \ldots, v_n]^T$ a column vector.

We use the notation:

$$P \times_\rho \mathbb{R}^n := P \times_G \mathbb{R}^n \text{ with } g \cdot v = \rho(g)v$$

Example 9.5. Take $\rho_0 : GL(r) \to GL(r)$ by $A \mapsto A$ the fundamental representation, and take its dual representation $\rho_0^* : GL(r) \to GL(r)$ given by $A \mapsto (A^{-1})^\dagger$. Then given $\pi : E \to M$ a vector bundle of rank r,

$$GL(E) \times_{\rho_0^{\otimes s} \otimes (\rho_0^*)^{\otimes t}} \mathbb{R}^{r^{s+t}} \cong E^{\otimes s} \otimes (E^*)^{\otimes t}$$

In particular, if M is a smooth manifold of dimension n, then:

$$GL(TM) \times \mathbb{R}^{n^{r+s}} \cong TM^{\otimes r} \otimes (T^*M)^{\otimes s} = T^r_s M$$

9.2 Cross Sections

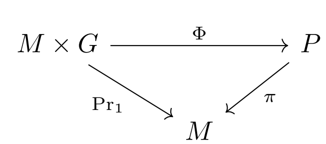
Definition 9.3. A **cross section** of a fiber bundle $\pi : E \to M$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. This means that for all $x \in M$, $\sigma(x) \in E_x$.

Lemma 9.1. *A principal G-bundle (π, P, M) is trivial if and only if it admits a cross section.*

Proof:

If $\pi : E \to M$ is a trivial fiber bundle, then it admits a cross section $\sigma(x) = \Psi^{-1}(x, \xi)$ where $\Psi : E \to M \times F$ is the bundle isomorphism with the product bundle and $\xi \in F$is chosen arbitrarily.

Conversely, if we let $\sigma : M \to P$ be a cross section, then we can define $\Phi : M \times G \to P$ by $\Phi(x, g) = \sigma(x) \cdot g$. Then Φ is a local diffeomorphism, and the diagram



commutes since:

$$\pi(\Phi(x, g)) = \pi(\sigma(x) \cdot g) = \pi(\sigma(x)) = x = \mathrm{Pr}_1(x, g)$$

Also, for all $x \in M$ and all $g, h \in G$, we have:

$$\Phi((x,h)\cdot g) = \Phi(x,hg) = \sigma(x)(hg) = (\sigma(x)h)g = \Phi(x,h)\cdot g$$

Which proves that (π, P, M) is the trivial G-bundle.

□

Lemma 9.2. $\sigma_\beta(x) = \sigma_\alpha(x)\cdot\psi_{\alpha\beta}(x)\forall x \in U_\alpha \cap U_\beta$

Proof:

$$\begin{aligned}
\sigma_\beta(x) &= \psi_\beta^{-1}(x,e) \\
&= \psi_\alpha^{-1}\circ\psi_\alpha\circ\psi_\beta^{-1}(x,e) \\
&= \psi_\alpha^{-1}(x,\psi_{\alpha\beta}(x)(e)) \\
&= \psi_\alpha^{-1}(x,e)\cdot\psi_{\alpha\beta}(x) \\
&= \sigma_\alpha(x)\cdot\psi_{\alpha\beta}(x)
\end{aligned}$$

□

9.3 Connections on a Principal Bundle

Definition 9.4. Given $\pi : E \to M$ a fiber bundle with fiber F, $\forall x \in M$, let $i_x : E_x \hookrightarrow E$ be the inclusion map. The **vertical space** V_u at $u \in E$ is the image of the injective linear map

$$(di_{\pi(u)})_u : T_u(E_{\pi(u)}) \to T_uE$$

For $\dim V_u = \dim F = N$, $\{V_u \mid u \in E\}$ is a C^∞ distribution of N-planes, which is to say that $V = \coprod_u v_u$ is a C^∞ sub-bundle of $TE \to E$ of rank N.

For example, if $\pi : E \to M$ is a vector bundle, then $V = \pi^*E$.

Lemma 9.3. *Given $\pi : P \to M$ a principal G-bundle, $V \cong P \times \mathfrak{g}$, where $\mathfrak{g} = T_eG$ is the Lie algebra of G.*

Definition 9.5. Given $\xi \in \mathfrak{g}$, the **fundamental vector field** $X_\xi^P \in \mathfrak{X}(P) = C^\infty(P, TP)$ is defined by

$$X_\xi^P(u) := \left.\frac{d}{dt}\right|_{t=0} u\cdot\exp(t\xi)$$

Recall that $\exp(t\xi) = \gamma(t)$, where γ is the integral curve of the left-invariant vector field $X_\xi^L \in \mathfrak{X}(G)$ defined by $X_\xi^L(e) = \xi$, and $\gamma(0) = e$. Now, $t \mapsto u\cdot\exp(t\xi)$ is a smooth curve in the fiber $P_{\pi(u)}$ over $\pi(u)$, passing through u at $t = 0$, which implies that $X_\xi^P(u) \in V_u \subset T_uP$. Indeed, $X_\xi^P \in C^\infty(P, V)$ is in the space of C^∞ sections of V.

Definition 9.6. Let $\pi : P \to M$ be a principal G-bundle, with $\dim M = n$ and $\dim G = N$. A **connection** on $\pi : P \to M$ is an assignment of a horizontal space $H_u \subset T_uP$ for each $u \in P$ such that $\{H_u \mid u \in P\}$ is a C^∞ distribution of n-planes, i.e, H is a C^∞ subbundle of rank n of TP, satisfying for all $u \in P$:

1. $T_uP = V_u \oplus H_u$
2. $H_{u\cdot a} = (dR_a)_u(H_u)$ for all $a \in G$, where $R_a : P \to P$ is the map $R_a(u) = u\cdot a$, which, as a smooth diffeomorphism, has derivative $(dR_a)_u : T_uP \to T_{u\cdot a}P$ given by a linear isomorphism.

Definition 9.7. A **connection 1-form** on a principal G-bundle $\pi : P \to M$ is a C^∞ $\mathfrak{g}$-valued 1-form ω ($\omega \in \Omega^1(P, \mathfrak{g})$, which is to say that $\forall X \in \mathfrak{X}(P)$, $\omega(X)$ is a smooth map from P to $\mathfrak{g}$) such that:

1. $\forall \xi \in \mathfrak{g}, \omega(X_\xi^P) = \xi$

2. $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega \quad \forall a \in G$ where $(R_a^*\omega)(X) = \omega((R_a)_*(X))$ as usual.

We want to show that these two definitions 9.6 and 9.7 are equivalent.

$(\implies)$

We know that by definition, $TP = V \oplus H$, so we have a projection $V \oplus H \to V$ which will define for us a smooth section ω' of:

$$T^*P \otimes V \cong T^*P \otimes (P \times \mathfrak{g})$$

Since the projection $V \oplus H \to V$ is exactly the map $TP \to P \times \mathfrak{g}$, this $\mathfrak{g}$-valued 1-form is defined by $(p, v) \mapsto (p, \omega'(v))$, and under the isomorphism $C^\infty(P, T^*P \otimes V) \cong \Omega^1(P, \mathfrak{g})$, we have that $\omega' \mapsto \omega$. Our goal is to prove that this ω is a connection 1-form.

Lemma 9.4. $\left((R_a)_* X_\xi^P\right)(u) = X^P_{\mathrm{Ad}(a^{-1})\xi}(u) \quad \forall a \in G, \quad \forall \xi \in \mathfrak{g}$

Proof:

$$\begin{aligned}
\left((R_a)_* X_\xi^P\right)(u) &= (dR_a)_{u\cdot a^{-1}}\left(X_\xi^P(u \cdot a^{-1})\right) \\
&= (dR_a)_{u\cdot a^{-1}}\left(\left.\frac{d}{dt}\right|_{t=0} u \cdot a^{-1}\exp(t\xi)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} u \cdot a^{-1}\exp(t\xi)a \\
&= \left.\frac{d}{dt}\right|_{t=0} u\exp(t\,\mathrm{Ad}(a^{-1})\xi) \\
&= X^P_{\mathrm{Ad}(a^{-1})\xi}(u)
\end{aligned}$$

Now, $\omega'(X_\xi^P) = X_\xi^P$ when ω' is considered as a projection, therefore $\omega(X_\xi^P) : P \to P \times \mathfrak{g}$ by $u \mapsto (u, \xi)$, which proves the first statement in Definition 9.7.

To prove the second part, it suffices to show that:

(1) $X \in \mathfrak{X}(P)$ such that $X(u) \in H_u \quad \forall u \in P$ implies $\omega((R_a)_*X) = \mathrm{Ad}(a^{-1})\omega(X)$

(2) $\forall \xi \in \mathfrak{g}$, $\omega((R_a)_*X_\xi^P) = \mathrm{Ad}(a^{-1})\omega(X_\xi^P)$

To prove (1), notice that both sides are just $0 \in \mathfrak{g}$, since $((R_a)_*X)(u) = (dR_a)_{u\cdot a^{-1}}\left(X(u\cdot a^{-1})\right) \in H_u$, because $X(u\cdot a^{-1}) \in H_{u\cdot a^{-1}}$, which implies $\omega((R_a)_*X) = 0$ if and only if $\omega(X) = 0$.

For (2), we use $\omega(X_\xi^P) = \xi$ to show:

$$\begin{aligned}
\omega((R_a)_*X_\xi^P) &= \omega(X^P_{\mathrm{Ad}(a^{-1})\xi}) \\
&= \mathrm{Ad}(a^{-1})\xi \\
&= \mathrm{Ad}(a^{-1})\omega(X_\xi^P)
\end{aligned}$$

$(\impliedby)$

Given $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the two conditions in Definition 9.7, $\forall u \in P$, $\omega(u) : T_uP \to \mathfrak{g}$. Define $H_u := \ker(\omega(u))$. Then the first condition of Definition 9.7 implies that $\omega(u)\big|_{V_u} : V_u \to \mathfrak{g}$ is a linear isomorphism, hence $T_uP = V_u \oplus H_u$, which gives condition 1 of Definition 9.6. To prove the second condition of Definition 9.6 from the second condition of Definition 9.7:

$$\omega(u \cdot a)(dR_a)_u = (R_a^*\omega)(u) = \mathrm{Ad}(a^{-1})\omega(u)$$

Which is the commutativity of:

$$\begin{array}{ccc} T_{u\cdot a}P & \xrightarrow{\ \omega(u\cdot a)\ } & \mathfrak{g} \\ \uparrow {\scriptstyle (dR_a)_u} & & \uparrow {\scriptstyle \mathrm{Ad}(a^{-1})} \\ T_uP & \xrightarrow[\ \omega(u)\ ]{} & \mathfrak{g} \end{array}$$

Since $v \in \ker(\omega(u))$ if and only if $(dR_a)_u(v) \in \ker(\omega(u \cdot a))$, we have that $(dR_a)_u(H_u) = H_{u\cdot a}$ as desired, which proves the equivalence of our definition of a connection on a principal bundle using horizontal spaces with the definition of using a connection 1-form.

Note: This connection 1 form ω is global.

Now, fix a principal G-bundle $\pi : P \to M$, and an open cover $\{U_\alpha\}_{\alpha \in I}$ of M together with local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ with local cross sections $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ given by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$. We want to consider all possible connection 1-forms $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying definition 9.7. Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the unique left-invariant $\mathfrak{g}$-valued 1-form on G such that:

$$\theta(e) = \mathrm{id} : T_eG \to T_eG = \mathfrak{g} \quad \text{and} \quad \theta(g) = \left(dL_{g^{-1}}\right)_g : T_gG \to T_eG = \mathfrak{g}$$

Example: For $G = GL(r, \mathbb{R}) \subset \mathbb{R}^{r^2}$ open, $A = (a_{ij})$, $dA = (da_{ij})$, $\theta = A^{-1}dA$

Note: For a general Lie group G, we may write $\theta = g^{-1}dg$. This θ is the unique $\mathfrak{g}$-valued 1-form on G such that $\theta(X_\xi^L) = \xi \quad \forall \xi \in \mathfrak{g}$. In fact, $X_\xi^L = X_\xi^G$ if we view G as the total space of a principal G-bundle over a point. Moreover, $\forall a \in G$, we have:

$$R_a^*\theta = R_a^*L_{a^{-1}}^*\theta = \mathrm{Ad}(a^{-1})\theta$$

Given any connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying definition 9.7, define:

$$\omega_\alpha := \sigma_\alpha^*\omega \in \Omega^1(U_\alpha, \mathfrak{g})$$

the pullback along the cross section $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$. For the inverse $\psi_\alpha^{-1} : U_\alpha \times G \to \pi^{-1}(U_\alpha)$ of our local trivialization, consider the pullback:

$$\left(\psi_\alpha^{-1}\right)^* \omega \in \Omega^1(U_\alpha \times G, \mathfrak{g})$$

which for the choice of some $x \in U_\alpha$ and $g \in G$ gives a map:

$$\left((\psi_\alpha^{-1})^*\omega\right)(x, g) : T_{(x,g)}(U_\alpha \times G) \to \mathfrak{g}$$

Note that for this product space, we have $T_{(x,g)}(U_\alpha \times G) = T_xU_\alpha \oplus T_gG$,

Notice that as maps $T_xU_\alpha \oplus T_gG \to \mathfrak{g}$, we have:

$$\left((\psi_\alpha^{-1})^*\omega\right)(x, g) = \left(\mathrm{Ad}(g^{-1}) \cdot \omega_\alpha(x)\right) \oplus \theta(g)$$

Lemma 9.5. *On $U_\alpha \cap U_\beta$, as elements of $\Omega^1(U_\alpha \cap U_\beta, \mathfrak{g})$, we have:*

$$\omega_\beta = \mathrm{Ad}\left(\psi_{\alpha\beta}^{-1}\right)\omega_\alpha + \psi_{\alpha\beta}^*\theta$$

Proof:

Recall the definitions of ψ_α and $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. We use Lemma 9.4 to conclude:

$$\begin{aligned}
\omega_\beta &= \sigma_\beta^*\omega \\
&= \sigma_\beta^*\psi_\alpha^*(\psi_\alpha^{-1})^*\omega \\
&= \sigma_\beta^*\psi_\alpha^*\left(\mathrm{Ad}(g^{-1})\omega_\alpha + \theta\right) \\
&= \left(\psi_\alpha \circ \sigma_\beta\right)^*\left(\mathrm{Ad}(g^{-1}\omega_\alpha + \theta)\right)
\end{aligned}$$

So that as a map, $\psi_\alpha \circ \sigma_\beta : U_\alpha \cap U_\beta \to (U_\alpha \cap U_\beta) \times G$ we have $(\psi_\alpha \circ \sigma_\beta)(x) = \psi_\alpha \circ \psi_\beta^{-1}(x, e) = (x, \psi_{\alpha\beta}(x))$

So that:

$$(\psi_\alpha \circ \sigma_\beta)^*\left(\mathrm{Ad}(g^{-1})\omega_\alpha + \theta\right) = \mathrm{Ad}(\psi_{\alpha\beta}^{-1})\omega_\alpha + \psi_{\alpha\beta}^*\theta$$

as desired. □

9.4 Connections on an Associated Vector Bundle

Give a $\pi : P \to M$ a principal $GL(r, \mathbb{F})$-bundle, take $\rho_0 : GL(r, \mathbb{F}) \to GL(r, \mathbb{F})$ the fundamental representation. We saw that $E = P \times_{\rho_0} \mathbb{F}^r$ is a smooth vector bundle of rank r, and that $GL(E) = P$. Let $\sigma_\alpha(x) = (e_{\alpha 1}(x), \ldots, e_{\alpha r}(x))$ be a local frame of E, which we can do since

$$\sigma_\alpha : U_\alpha \to P|_{U_\alpha} = GL(E)\big|_{U_\alpha}$$

Now, each $e_{\alpha,i} : U_\alpha \to E_{U_\alpha}$ is a smooth section of $E|_{U_\alpha}$, and each $(e_{\alpha 1}(x), \ldots, e_{\alpha r}(x))$ is a basis of E_x. Given a connection on P, we define ∇ on E as follows:

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$

is an $\mathbb{F}$-linear map sending $S \mapsto \nabla S$, such that

$$\nabla(fS) = df \otimes S + f\nabla S$$

For all $f \in C^\infty(M)$ and $S \in \Omega^0(M, E)$. Writing $\omega_\alpha = \sigma_\alpha^*\omega = \left(\theta_{ij}\right)_{i,j=1}^r$, for $\theta_{ij} \in \Omega^1(U_\alpha)$ as above. On $E|_{U_\alpha}$ define $\nabla e_{\alpha,i} = \sum_j e_{\alpha,j} \otimes \theta_{ji}$ which is to say:

$$\begin{bmatrix}\nabla_{e_{\alpha_1}} & \cdots & \nabla_{e_{\alpha_r}}\end{bmatrix} = \begin{bmatrix}e_{\alpha_1} & \cdots & e_{\alpha_r}\end{bmatrix}\begin{bmatrix}\theta_1^1 & \cdots & \theta_r^1 \\ \vdots & \ddots & \vdots \\ \theta_1^r & \cdots & \theta_r^r\end{bmatrix}$$

As helpful notation, we will write the above as:

$$\nabla\sigma_\alpha = \sigma_\alpha\omega_\alpha$$

We have also seen that:

$$\omega_\beta = \sigma_\alpha\psi_{\alpha\beta}$$

a shorthand of:

$$\begin{bmatrix} e_{\beta_1} & \cdots & e_{\beta_r} \end{bmatrix} = \begin{bmatrix} e_{\alpha_1} & \cdots & e_{\alpha_r} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix}$$

So:

$$\begin{aligned} \nabla\sigma_\beta &= (\nabla\sigma_\alpha)\psi_{\alpha\beta} + \sigma_\alpha d\psi_{\alpha\beta} \\ &= \sigma_\alpha\left(\omega_\alpha\psi_{\alpha\beta} + d\psi_{\alpha\beta}\right) \\ &= \sigma_\beta\left(\psi_{\alpha\beta}^{-1}\omega_\alpha\psi_{\alpha\beta} + \psi_{\alpha\beta}^{-1}d\psi_{\alpha\beta}\right) \\ &= \sigma_\beta\omega_\beta \end{aligned}$$

Which checks that the formula transforms correctly.

Given $s \in C^\infty(M, E)$, on U_α we can write s locally as $s = \sum_{i=1}^{r} s_{\alpha i}e_{\alpha i}$, or:

$$s = \begin{bmatrix} e_{\alpha 1} & \cdots & e_{\alpha r} \end{bmatrix} \begin{bmatrix} s_{\alpha 1} \\ \vdots \\ s_{\alpha r} \end{bmatrix}$$

And use the notation to collapse this to $s = \sigma_\alpha s_\alpha$

Then, for $\nabla s \in \Omega^1(M, E)$ on U_α, it takes the form:

$$\begin{aligned} \nabla s &= \nabla(\sigma_\alpha s_\alpha) \\ &= (\nabla\sigma_\alpha)s_\alpha + \sigma_\alpha ds_\alpha \\ &= \sigma_\alpha\omega_\alpha s_\alpha + \sigma_\alpha ds_\alpha \\ &= \sigma_\alpha\left(\omega_\alpha s_\alpha + ds_\alpha\right) \end{aligned}$$

Which allows us to conclude that $(\nabla s)_\alpha = \omega_\alpha s_\alpha + ds_\alpha$

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} = \begin{bmatrix} d(s_1) \\ \vdots \\ d(s_r) \end{bmatrix} + \begin{bmatrix} \theta_1^1 & \cdots & \theta_r^1 \\ \vdots & \ddots & \vdots \\ \theta_1^r & \cdots & \theta_r^r \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix}$$

In general, given a principal G-bundle $\pi : P \to M$ and a representation $\rho : G \to GL(m, \mathbb{F})$, let $E = P \times_\rho \mathbb{F}$ be the associated vector bundle of rank m. With our usual notation for transition functions of P, the transition functions of E are given by:

$$\begin{aligned} (U_\alpha \cap U_\beta) \times \mathbb{F}^m &\to (U_\alpha \cap U_\beta) \times \mathbb{F}^m \\ (x, v) &\mapsto (x, \rho(\varphi_{\alpha\beta}(x))v) \end{aligned}$$

Let $\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ be defined by a connection on P so that $\omega_\beta = \mathrm{Ad}(\psi_{\alpha\beta}^{-1})\omega_\alpha + \psi_{\alpha\beta}^*\theta$.

Consider the corresponding Lie algebra representation:

$$(d\rho)_e : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{F})$$

and define

$$\omega_\alpha^\rho := (d\rho_e)\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(m, \mathbb{F}))$$

Then we know that this satisfies:

$$\omega_\beta^\rho = (\rho \circ \phi_{\alpha\beta})^{-1}\omega_\alpha(\rho \circ \psi_{\alpha\beta}) + (\rho \circ \psi_{\alpha\beta})^{-1}d(\rho \circ \psi_{\alpha\beta})$$

So, $\{\omega_\alpha^\rho \mid \alpha \in I\}$ defines a connection $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ on the vector bundle.

9.5 Horizontal Lifts

Given $\pi : P \to M$ a principal G-bundle with connection $\Gamma = \{H_u \subset T_uP \mid u \in P\}$, recall that the map $d\pi_u|_{H_u} : H_u \to T_{\pi(u)}M$ is a linear isomorphism.

Definition 9.8. Given $X \in \mathfrak{X}(M)$, the **horizontal lift** $X^* \in \mathfrak{X}(P)$ of X is defined as follows.

$$\forall u \in P, X^*(u) \in H_u \quad \text{and} \quad d\pi_u(X^*(u)) = X(\pi(u))$$

Indeed, the resulting vector field on P is horizontal, so we write $X^* \in C^\infty(P, H)$. Note that $(R_aX)_*X^* = X^*$ for all $a \in G$, since $(dR_a)_u(X^*(u)) = X^*(u \cdot a)$ which comes from the fact that $(dR_a)_u(H_u) = H_{u \cdot a}$. We have an injective $\mathbb{R}$-linear map

$$\mathfrak{X}(M) \to C^\infty(P, H)$$

by $X \mapsto X^*$. The image of this map is:

$$\{\tilde{X} \in C^\infty(P, H) \mid (R_a)_*\tilde{X} = \tilde{X} \quad \forall a \in G\}$$

Given any such $\tilde{X}$, define $X(x) = (d\pi_u)(\tilde{X}(u))$ for any $u \in \pi^{-1}(x)$. Then $\tilde{X} = X^*$

Note also that for any $\tilde{X} \in \mathfrak{X}(P)$, there exist unique components $\tilde{X}^V \in C^\infty(P, V)$ and $\tilde{X}^H \in C^\infty(P, H)$ such that $\tilde{X} = \tilde{X}^V + \tilde{X}^H$.

Lemma 9.6. *Given $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have:*

1. $(X + Y)^* = X^* + Y^*$
2. $(fX)^* = (f \circ \pi)X^*$

3. $[X,Y]^* = [X^*, Y^*]^H$

Now, let $\pi : P \to M$ be a principal G-bundle with connection $\Gamma = \{H_u \subset T_uP \mid u \in P\}$, and consider a smooth curve $\alpha : [a,b] \to M$.

Definition 9.9. A **horizontal lift** of α is a smooth curve $\tilde{\alpha} : [a,b] \to P$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}'(t) \in H_{\alpha(t)} \quad \forall t \in [a,b]$

9.6 Parallel Transport

Let $\pi : P \to M$ be a principal G-bundle with connection $\Gamma = \{H_u \subset T_uP \mid u \in P\}$, and let $\gamma : [a,b] \to M$ be a piecewise smooth curve. The **parallel transport along** γ is the map:

$$\mathrm{Hol}(\gamma) : \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(b))$$

sending $u \mapsto \tilde{\gamma}(b)$ where $\tilde{\gamma}$ is the unique horizontal lift of γ, satisfying $\widetilde{\gamma(a)} = u$.

Here are some properties of this parallel transport definition:

1. Reparametrizing $\phi : [c,d] \to [a,b], \phi' \geq 0$ implies $\mathrm{Hol}(\gamma) = \mathrm{Hol}(\gamma \circ \phi)$
2. $\mathrm{Hol}(\gamma_2 \cdot \gamma_1) = \mathrm{Hol}(\gamma_2) \circ \mathrm{Hol}(\gamma_1)$
3. γ constant implies $\mathrm{Hol}(\gamma) = \mathrm{Id} : \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(a))$
4. $\mathrm{Hol}(\gamma^{-1}) = (\mathrm{Hol}(\gamma))^{-1} : \pi^{-1}(\gamma(b)) \to \pi^{-1}(\gamma(a))$
5. $\forall u \in \pi^{-1}(\gamma(a))$ and $\forall g \in G$, $\mathrm{Hol}(\gamma)(u \cdot g) = (\mathrm{Hol}(\gamma)(u)) \cdot g$

Definition 9.10. Given M_1, M_2 smooth manifolds both admitting right actions by a Lie group G, a smooth map $f : M_1 \to M_2$ is G-equivariant if $f(x \cdot g) = f(x) \cdot g \ \forall x \in M, \forall g \in G$.

Therefore, property 5 above states that $\mathrm{Hol}(\gamma)$ is always a G-equivariant map between fibers.

Definition 9.11. Let $\pi : P \to M$ be a principal G-bundle with connection $\Gamma = \{H_u \subset T_uP \mid u \in P\}$. For a fixed $x \in M$, the set of piecewise smooth curves $\gamma : [0,1] \to M$ satisfying $\gamma(0) = \gamma(1) = x$ is called the **loop space** $\Omega(M,x)$ based at $x \in M$.

If we consider the connected component

$$\Omega^0(M,x) = \{\gamma \in \Omega(M,x) \mid \gamma \sim \gamma_0\}$$

of the loop space based at x consisting of curves homotopic to the constant curve $\gamma_0 : [0,1] \to M$ sending $t \mapsto x$ for all $t \in [0,1]$. Let $\pi_1(M,x)$ denote the quotient $\Omega(M,x)/\Omega^0(M,x)$. Now, for all $\gamma \in \Omega(M,x)$, $\mathrm{Hol}(\gamma) : \pi^{-1}(x) \to \pi^{-1}(x)$.

Definition 9.12. The collection

$$\Phi(x) = \{\mathrm{Hol}(\gamma) \mid \gamma \in \Omega(M,x)\}$$

forms a group known as the **holonomy group of** Γ **with reference point** x. We also can define the **restricted holonomy group**:

$$\Phi^0(x) = \{\mathrm{Hol}(\gamma) \mid \gamma \in \Omega^0(M,x)\}$$

Now, pick a point $u \in \pi^{-1}(x)$, and define $\phi_u : \Omega(M,x) \to G$ by:

$$\mathrm{Hol}(\gamma)(u) = u \cdot \phi_u(\gamma)$$

Which we can do because G acts transitively on the fibers of a principal G-bundle.

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